

ON TERMINAL FANO 3-FOLDS WITH 2-TORUS ACTION

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ABSTRACT. We classify the terminal \mathbb{Q} -factorial Fano threefolds of Picard number one that come with an effective action of a two-dimensional torus. Our approach applies also to higher dimensions and generalizes the correspondence between toric Fano varieties and lattice polytopes: to any Fano variety with a complete intersection Cox ring we associate its “anticanonical complex”, which is a certain polyhedral complex living in the lattice of one parameter groups of an ambient toric variety. For resolutions constructed via the tropical variety, the lattice points inside the anticanonical complex control the discrepancies. This leads, for example, to simple characterizations of terminality and canonicity.

1. THE MAIN RESULTS

This article contributes to the classification of Fano threefolds, that means normal projective algebraic varieties X of dimension three with an ample anticanonical divisor; we work over the field \mathbb{C} of complex numbers. Whereas the smooth Fano threefolds are well known due to Iskovskikh [9, 10] and Mori/Mukai [15], the singular case is still widely open. We restrict to terminal singularities, i.e., the mildest class in the context of the minimal model program. Let $\mathbb{T} \subseteq \text{Aut}(X)$ be a maximal torus. If $\dim(\mathbb{T}) = \dim(X)$ holds, then X is a toric Fano variety and the classification can be performed in the setting of lattice polytopes, see [3, 12]. We go one step further and consider torus actions of complexity one, meaning that we have $\dim(\mathbb{T}) = \dim(X) - 1$. Our approach is via the Cox ring

$$\mathcal{R}(X) := \bigoplus_{\text{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$

which can be associated to any normal complete variety X with finitely generated divisor class group $\text{Cl}(X)$; see [1, Sec. 1.4] for the details of this definition. For a Fano variety X with at most terminal singularities, $\text{Cl}(X)$ is finitely generated [11, Sec. 2.1]. If, in addition, X comes with a torus action of complexity one, then X is rational, the Cox ring $\mathcal{R}(X)$ is finitely generated, uniquely determines X , and admits an explicit description as a complete intersection [5, 7]. Our main result gives a classification of the terminal \mathbb{Q} -factorial threefolds of Picard number one by listing their Cox rings.

Theorem 1.1. *The following table lists the Cox rings $\mathcal{R}(X)$ of the non-toric terminal \mathbb{Q} -factorial Fano threefolds X of Picard number one with an effective two-torus action; the $\text{Cl}(X)$ -degrees of the generators T_1, \dots, T_r are denoted as columns $w_i \in \text{Cl}(X)$ of a matrix $[w_1, \dots, w_r]$. Additionally we give the selfintersection number $(-\mathcal{K}_X)^3$ for the anticanonical class $-\mathcal{K}_X \in \text{Cl}(X)$ and the Gorenstein index $\iota(X)$, i.e., the smallest positive integer such that $\iota(X) \cdot \mathcal{K}_X$ is Cartier.*

No.	$\mathcal{R}(X)$	$\text{Cl}(X)$	$[w_1, \dots, w_r]$	$(-\mathcal{K}_X)^3$	$\iota(X)$
1	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3 T_4 + T_5^2 \rangle}$	\mathbb{Z}	$[1 \ 1 \ 1 \ 1 \ 1]$	54	1

2	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3 T_4 + T_5^2 \rangle}$	\mathbb{Z}	$[1 \ 5 \ 2 \ 4 \ 3]$	729/20	20
3	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3 T_4 + T_5^2 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$	$[\frac{1}{2} \ \frac{1}{3} \ \frac{1}{1} \ \frac{1}{4} \ \frac{1}{0}]$	54/5	5
4	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3 T_4 + T_5^3 \rangle}$	\mathbb{Z}	$[1 \ 5 \ 3 \ 3 \ 2]$	512/15	15
5	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3 T_4 + T_5^4 \rangle}$	\mathbb{Z}	$[1 \ 3 \ 2 \ 2 \ 1]$	125/3	6
6	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3 T_4 + T_5^4 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\frac{1}{1} \ \frac{3}{1} \ \frac{2}{1} \ \frac{2}{1} \ \frac{1}{0}]$	125/6	12
7	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3 T_4 + T_5^6 \rangle}$	\mathbb{Z}	$[2 \ 4 \ 3 \ 3 \ 1]$	343/12	12
8	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^2 T_4 + T_5^3 \rangle}$	\mathbb{Z}	$[1 \ 3 \ 1 \ 2 \ 2]$	125/3	6
9	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^2 T_4 + T_5^2 \rangle}$	\mathbb{Z}	$[1 \ 5 \ 2 \ 2 \ 3]$	343/10	10
10	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^2 T_4 + T_5^2 \rangle}$	\mathbb{Z}	$[3 \ 7 \ 4 \ 2 \ 5]$	1331/84	84
11	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^2 T_4 + T_5^3 \rangle}$	\mathbb{Z}	$[2 \ 1 \ 1 \ 1 \ 1]$	81/2	2
12	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^2 T_4 + T_5^3 \rangle}$	\mathbb{Z}	$[3 \ 3 \ 1 \ 4 \ 2]$	343/12	12
13	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^2 T_4 + T_5^3 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$[\frac{2}{1} \ \frac{1}{2} \ \frac{1}{1} \ \frac{1}{1} \ \frac{1}{0}]$	27/2	6
14	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^2 T_4 + T_5^6 \rangle}$	\mathbb{Z}	$[3 \ 3 \ 2 \ 2 \ 1]$	125/6	6
15	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^2 T_4^2 + T_5^2 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\frac{1}{1} \ \frac{3}{1} \ \frac{1}{0} \ \frac{1}{0} \ \frac{2}{1}]$	64/3	6
16	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^2 T_4^2 + T_5^3 \rangle}$	\mathbb{Z}	$[3 \ 3 \ 2 \ 1 \ 2]$	125/6	6
17	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 T_4 + T_5^2 \rangle}$	\mathbb{Z}	$[1 \ 3 \ 1 \ 1 \ 2]$	128/3	3
18	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 T_4 + T_5^2 \rangle}$	\mathbb{Z}	$[2 \ 4 \ 1 \ 3 \ 3]$	343/12	12
19	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 T_4 + T_5^2 \rangle}$	\mathbb{Z}	$[3 \ 7 \ 2 \ 4 \ 5]$	1331/84	84
20	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 T_4 + T_5^2 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\frac{1}{1} \ \frac{3}{1} \ \frac{1}{0} \ \frac{1}{0} \ \frac{2}{1}]$	64/3	6
21	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 T_4 + T_5^4 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\frac{2}{1} \ \frac{2}{1} \ \frac{1}{1} \ \frac{1}{1} \ \frac{1}{0}]$	27/2	4
22	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 T_4^2 + T_5^2 \rangle}$	\mathbb{Z}	$[3 \ 5 \ 2 \ 1 \ 4]$	343/15	30
23	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 T_4^2 + T_5^2 \rangle}$	\mathbb{Z}	$[2 \ 4 \ 1 \ 1 \ 3]$	125/4	4
24	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^5 T_4 + T_5^2 \rangle}$	\mathbb{Z}	$[2 \ 4 \ 1 \ 1 \ 3]$	125/4	4
25	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^6 T_4 + T_5^2 \rangle}$	\mathbb{Z}	$[3 \ 5 \ 1 \ 2 \ 4]$	343/15	30
26	$\frac{\mathbb{C}[T_1, \dots, T_6]}{\langle T_1 T_2 + T_3 T_4 + T_5^2, \lambda T_3 T_4 + T_5^2 + T_6^2 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\frac{1}{1} \ \frac{1}{1} \ \frac{1}{0} \ \frac{1}{0} \ \frac{1}{1} \ \frac{1}{0}]$	16	2
27	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 T_3 + T_4^3 + T_5^2 \rangle}$	\mathbb{Z}	$[1 \ 1 \ 4 \ 2 \ 3]$	125/4	4
28	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 T_3 + T_4^3 + T_5^2 \rangle}$	\mathbb{Z}	$[2 \ 3 \ 1 \ 2 \ 3]$	125/6	6
29	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 + T_4^2 \rangle}$	\mathbb{Z}	$[1 \ 5 \ 2 \ 3 \ 1]$	216/5	5
30	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 + T_4^2 \rangle}$	\mathbb{Z}	$[1 \ 5 \ 2 \ 3 \ 2]$	343/10	10
31	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 + T_4^2 \rangle}$	\mathbb{Z}	$[1 \ 5 \ 2 \ 3 \ 3]$	512/15	15
32	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 + T_4^2 \rangle}$	\mathbb{Z}	$[1 \ 5 \ 2 \ 3 \ 4]$	729/20	20
33	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^4 + T_4^2 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\frac{1}{1} \ \frac{3}{1} \ \frac{1}{0} \ \frac{2}{1} \ \frac{1}{0}]$	64/3	6
34	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^4 + T_4^2 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$[\frac{1}{1} \ \frac{3}{1} \ \frac{1}{0} \ \frac{2}{1} \ \frac{2}{1}]$	125/6	12

35	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^5 + T_4^2 \rangle}$	\mathbb{Z}	$[3 \ 7 \ 2 \ 5 \ 1]$	512/21	21
36	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^5 + T_4^2 \rangle}$	\mathbb{Z}	$[3 \ 7 \ 2 \ 5 \ 4]$	1331/84	84
37	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^6 + T_4^2 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\begin{bmatrix} 2 & 4 & 1 & 3 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$	125/8	8
38	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^3 + T_4^3 \rangle}$	$\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$	$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{bmatrix}$	27/2	6
39	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^4 + T_4^3 \rangle}$	\mathbb{Z}	$[5 \ 7 \ 3 \ 4 \ 1]$	512/35	35
40	$\frac{\mathbb{C}[T_1, \dots, T_5]}{\langle T_1 T_2 + T_3^4 + T_4^3 \rangle}$	\mathbb{Z}	$[5 \ 7 \ 3 \ 4 \ 2]$	729/70	70

where $\lambda \in \mathbb{C}^* \setminus \{1\}$ in No. 26. Any two of the Cox rings $\mathcal{R}(X)$ listed in the table correspond to non-isomorphic varieties. All corresponding varieties X are rational and No. 1 is the only smooth one.

Our approach works also in higher dimensions and applies more generally to Fano varieties X with a complete intersection Cox ring $\mathcal{R}(X)$. For such varieties we introduce the anticanonical complex as a combinatorial data in the spirit of the Fano polytopes from toric geometry. Theorem 1.4 characterizes in particular canonical and terminal singularities in terms of lattice points inside the anticanonical complex. Using the knowledge on the Cox ring of varieties with a torus action of complexity one provided by [7, 5], we obtain an explicit description of the anticanonical complex in that case, see Section 4. This enables us to derive in Section 5 effective bounds for the defining data of the terminal Fano threefolds of Picard number one that come with an action of a two-dimensional torus. One of the basic principles is to construct suitable lattice simplices via the anticanonical complex and to apply the volume bounds obtained in [2, 12, 13]. Having found reasonable bounds, the remaining step is to figure out the terminal ones from the list of possible candidates by means of Theorem 1.4. This is done using the software package [6], where among other things our methods are implemented, and finally leads to the list given in Theorem 1.1.

We now present the construction of the anticanonical complex and the characterization of singularities. Consider a normal Fano variety X with divisor class group $\text{Cl}(X)$ and Cox ring $\mathcal{R}(X)$. Recall that $\mathcal{R}(X)$ is factorially $\text{Cl}(X)$ -graded, i.e., every homogeneous nonzero nonunit is a product of $\text{Cl}(X)$ -primes, see [4, Sec. 3]. We assume that $\mathcal{R}(X)$ is a complete intersection in the sense that it comes with a presentation by $\text{Cl}(X)$ -homogeneous generators T_ϱ and relations g_i :

$$\mathcal{R}(X) = \mathbb{C}[T_\varrho; \varrho \in \mathbf{R}] / \langle g_1, \dots, g_s \rangle,$$

where the meaning of the index set \mathbf{R} becomes clear soon, the T_ϱ define pairwise nonassociated $\text{Cl}(X)$ -primes in $\mathcal{R}(X)$ and the dimension of $\mathcal{R}(X)$ equals $|\mathbf{R}| - s$. This setting leads to a closed embedding $X \subseteq Z_\Sigma$ into a toric variety Z_Σ arising from a fan Σ , where the divisor class group and Cox ring of Z_Σ are given by

$$\text{Cl}(Z_\Sigma) \cong \text{Cl}(X), \quad \mathcal{R}(Z_\Sigma) = \mathbb{C}[T_\varrho; \varrho \in \mathbf{R}];$$

see [4, Constr. 3.13 and Prop. 3.14]. Removing successively closed torus orbits from Z_Σ , we can achieve that X intersects every closed torus orbit of Z_Σ . We speak then of $X \subseteq Z_\Sigma$ as the minimal toric embedding.

We provide the necessary details for defining the anticanonical complex. Consider the degree homomorphism $Q: \mathbb{Z}^{\mathbf{R}} \rightarrow \text{Cl}(X)$ sending the ϱ -th canonical basis vector $e_\varrho \in \mathbb{Z}^{\mathbf{R}}$ to $\deg(T_\varrho) \in \text{Cl}(X)$ and let $P^*: \mathbb{Z}^n \rightarrow \mathbb{Z}^{\mathbf{R}}$ be a linear embedding with image $\ker(Q)$. Then we have

$$\text{Cl}(Z_\Sigma) \cong \mathbb{Z}^{\mathbf{R}} / P^*(\mathbb{Z}^n) \cong \text{Cl}(X).$$

Denote by $P: \mathbb{Z}^R \rightarrow \mathbb{Z}^n$ the dual map of P^* . Set $e_\Sigma := \sum e_\varrho$. Then the canonical classes of Z_Σ and X are given as

$$\mathcal{K}_\Sigma = -Q(e_\Sigma), \quad \mathcal{K}_X = \sum \deg(g_i) + \mathcal{K}_\Sigma.$$

The defining fan Σ of Z_Σ lives in the lattice \mathbb{Z}^n and is obtained as follows. Let $\gamma_R \subseteq \mathbb{Q}^R$ be the positive orthant, spanned by the e_ϱ , and $e_X \in \mathbb{Z}^R$ any representative of \mathcal{K}_X . Then we have polytopes

$$B(-\mathcal{K}_X) := Q^{-1}(-\mathcal{K}_X) \cap \gamma_R \subseteq \mathbb{Q}^R, \quad (P^*)^{-1}(B(-\mathcal{K}_X) + e_X) \subseteq \mathbb{Q}^n.$$

The normal fan Σ_c of the second polytope defines a toric variety Z_c containing X as a subvariety and Σ is the subfan of Σ_c generated by the cones that correspond to a torus orbit of Z_c intersecting X . In particular, the rays of Σ have exactly the vectors $v_\varrho := P(e_\varrho) \in \mathbb{Z}^n$ as their primitive generators; we identify $\varrho \in R$ with the ray through v_ϱ .

Let $\text{trop}(X) \subseteq \mathbb{Q}^n$ denote the tropical variety of $X \cap \mathbb{T}$, endowed with a fan structure that refines the normal fan $\mathcal{N}(B)$ of the Minkowski sum $B := B(g_1) + \dots + B(g_s)$ of the Newton polytopes $B(g_i)$ of the relations g_i , i.e., $B(g_i) \subseteq \mathbb{Q}^R$ is the convex hull over the exponent vectors of g_i .

Definition 1.2. The *anticanonical polyhedron* of X is the dual polyhedron $A_X \subseteq \mathbb{Q}^n$ of the polytope

$$B_X := (P^*)^{-1}(B(-\mathcal{K}_X) + B - e_\Sigma) \subseteq \mathbb{Q}^n.$$

The *anticanonical complex* of X is the coarsest common refinement of polyhedral complexes

$$A_X^c := \text{faces}(A_X) \sqcap \Sigma \sqcap \text{trop}(X).$$

The *relative interior* of A_X^c is the interior of its support with respect to the tropical variety $\text{trop}(X)$.

Example 1.3. The E_6 -singular cubic surface $X = V(z_1 z_2^2 + z_2 z_0^2 + z_3^3) \subseteq \mathbb{P}_3$ is invariant under the \mathbb{C}^* -action

$$t \cdot [z_0, \dots, z_3] = [z_0, t^{-3} z_1, t^3 z_2, t z_3]$$

on \mathbb{P}_3 . The divisor class group and the Cox ring of the surface X are explicitly given by

$$\text{Cl}(X) = \mathbb{Z}, \quad \mathcal{R}(X) = \mathbb{C}[T_1, T_2, T_3, T_4] / \langle T_1 T_2^3 + T_3^3 + T_4^2 \rangle,$$

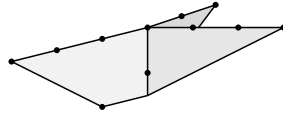
where the $\text{Cl}(X)$ -degrees of T_1, T_2, T_3, T_4 are 3, 1, 2, 3. The minimal ambient toric variety Z_Σ is an open subset of $Z_c = \mathbb{P}_{3,1,2,3}$ and the tropical variety in \mathbb{Q}^3 is

$$\text{trop}(X) = \text{cone}(e_1, \pm e_3) \cup \text{cone}(e_2, \pm e_3) \cup \text{cone}(-e_1 - e_2, \pm e_3),$$

where $e_i \in \mathbb{Q}^3$ is the i -th canonical basis vector. The anticanonical polyhedron $A_X \subseteq \mathbb{Q}^3$ has the vertices

$$(-3, -3, -2), (-1, -1, -1), (3, 0, 1), (0, 2, 1), (0, 0, 1), (0, 0, -1/5).$$

The anticanonical complex $A_X^c = A_X \sqcap \text{trop}(X)$ lives on the three cones of $\text{trop}(X)$ and thus is of dimension two.



Our aim is to characterize the behaviour of singularities of X in terms of lattice points of the anticanonical complex A_X^c . Recall that for a \mathbb{Q} -Gorenstein variety X , that means that some non-zero multiple of the canonical divisor \mathcal{K}_X is Cartier, various types of singularities are defined via the ramification formula

$$\mathcal{K}_{X'} - \varphi^*(\mathcal{K}_X) = \sum a_i E_i,$$

where $\varphi: X' \rightarrow X$ is a resolution, the E_i are the prime components of the exceptional divisor and the a_i are called the *discrepancies* of the resolution. One says that X has at most *log terminal* (ε -*log terminal* for $0 < \varepsilon < 1$, *canonical*, *terminal*) singularities, if for every resolution the discrepancies a_i satisfy $a_i > -1$ ($a_i > -1 + \varepsilon$, $a_i \geq 0$, $a_i > 0$).

We concern ourselves with Fano varieties X that are *strongly tropically resolvable* in the sense that any subdivision of $\Sigma \sqcap \text{trop}(X)$ admits a regular refinement that induces a resolution of singularities $X' \rightarrow X$ with a Mori dream space X' . As we will see in Proposition 3.7, all normal rational varieties with a torus action of complexity one are strongly tropically resolvable.

Theorem 1.4. *Let X be a (strongly tropically resolvable) normal Fano variety with a complete intersection Cox ring.*

- (i) A_X^c contains the origin in its relative interior and all primitive generators of the fan Σ are vertices of A_X^c .
- (ii) X has at most log terminal singularities if (and only if) the anticanonical complex A_X^c is bounded.
- (iii) X has at most ε -log terminal singularities if (and only if) 0 is the only lattice point in εA_X^c .
- (iv) X has at most canonical singularities if (and only if) 0 is the only lattice point in the relative interior of A_X^c .
- (v) X has at most terminal singularities if (and only if) 0 and the primitive generators v_ϱ for $\varrho \in \Sigma^{(1)}$ are the only lattice points of A_X^c .

Note that these statements generalize the corresponding characterizations of toric singularities in terms of lattice polytopes given for example in [3]. In the toric case, i.e., in the absence of relations g_i , our anticanonical polytope A_X is just the Fano polytope and the anticanonical complex is the subdivision of A_X by the fan Σ .

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2. DISCREPANCIES

Here we prove Theorem 1.4. The setting is the one introduced in Section 1. In particular, X is a normal Fano variety with a complete intersection Cox ring $\mathcal{R}(X)$ given by $\text{Cl}(X)$ -homogeneous generators and relations and we have the associated minimal toric embedding:

$$\mathcal{R}(X) = \mathbb{C}[T_\varrho; \varrho \in \mathbb{R}] / \langle g_1, \dots, g_s \rangle, \quad X \subseteq Z_\Sigma.$$

Assertion (i) of Theorem 1.4 holds under more general assumptions. Therefore, we state and prove it separately. Note that we always have $0 \in \text{reint}(A_X)$. For any ray $\varrho \in \mathbb{R}$ with $\varrho \not\subseteq A_X$ we denote by v'_ϱ the intersection point of ϱ and the boundary ∂A_X .

Proposition 2.1. *Assume that the anticanonical class $-\mathcal{K}_X$ lies in the relative interior of the movable cone of X . Then, for every ray $\varrho \in \Sigma$, the primitive generator $v_\varrho \in \varrho$ is a vertex of A_X . In particular, $\varrho \not\subseteq A_X$ and we have $v'_\varrho = v_\varrho$.*

Proof. By construction, the anticanonical polyhedron A_X is the intersection of the half spaces

$$H_u := \{v \in \mathbb{Q}^n; \langle u, v \rangle \geq -1\}, \quad u \in B_X.$$

Thus, our task is to show that for every ray $\varrho \in \mathbb{R}$ there is a facet $B_\varrho \subseteq B_X$ with $\langle b, v_\varrho \rangle = -1$ for all $b \in B_\varrho$.

Fix $\varrho \in \mathbb{R}$. Then $v_\varrho = P(e_\varrho)$ holds with a unique canonical basis vector $e_\varrho \in \mathbb{Z}^{\mathbb{R}}$. Thus, for any $u \in \mathbb{Q}^n$, we have

$$\langle u, v_\varrho \rangle = \langle u, P(e_\varrho) \rangle = \langle P^*(u), e_\varrho \rangle$$

and $P^*(B_X)$ equals $B(-\mathcal{K}_X) + B - e_\Sigma$. Since $B(-\mathcal{K}_X)$ and B both lie in the positive orthant $\gamma_{\mathbb{R}}$, we conclude $\langle u, v_\varrho \rangle \geq -1$ for all $u \in B_X$.

Let $\gamma_\varrho \preceq \gamma_{\mathbb{R}}$ be the facet consisting of points with ϱ -th coordinate zero. The description of the movable cone given in [4, Prop. 4.1] shows that $-\mathcal{K}_X$ lies in the relative interior of $Q(\gamma_\varrho)$. It follows that

$$B^\varrho(-\mathcal{K}_X) := B(-\mathcal{K}_X) \cap \gamma_\varrho$$

is a facet of $B(-\mathcal{K}_X)$. Note that $\langle e, e_\varrho \rangle = 0$ holds for all $e \in B^\varrho(-\mathcal{K}_X)$. We claim that $B^\varrho := B \cap \gamma_\varrho$ is nonempty. Indeed, since every g_i is irreducible, it has an exponent $b_i \in B(g_i)$ with ϱ -th coordinate zero. Thus $b_1 + \dots + b_s \in B^\varrho$ holds. Note that we have $\langle e, e_\varrho \rangle = 0$ for all $e \in B^\varrho$. Since zero is the minimal possible value for linear forms from $B(-\mathcal{K}_X) + B$ on e_ϱ , we see that $B^\varrho(-\mathcal{K}_X) + B^\varrho - e_\Sigma$ is a face of $B(-\mathcal{K}_X) + B - e_\Sigma$. By dimension reasons, it is a facet. \square

Consider a toric modification $Z_{\Sigma'} \rightarrow Z_\Sigma$ given by a subdivision $\Sigma' \rightarrow \Sigma$ of fans. We introduce a shift of polynomials from $\mathbb{C}[T_\varrho; \varrho \in \mathbb{R}]$ to $\mathbb{C}[T_{\varrho'}; \varrho' \in \mathbb{R}']$, where $\mathbb{R} \subseteq \Sigma$ and $\mathbb{R}' \subseteq \Sigma'$ are the sets of rays. The toric Cox constructions $P: \mathbb{Z}^{\mathbb{R}} \rightarrow \mathbb{Z}^n$ and $P': \mathbb{Z}^{\mathbb{R}'} \rightarrow \mathbb{Z}^n$ define homomorphisms of tori

$$\mathbb{T}^{\mathbb{R}'} \xrightarrow{P'} \mathbb{T}^n \xleftarrow{P} \mathbb{T}^{\mathbb{R}}.$$

Let $g \in \mathbb{C}[T_\varrho; \varrho \in \mathbb{R}]$ be without monomial factors. The push-down of g is the unique $p_*(g) \in \mathbb{C}[T_1, \dots, T_n]$ without monomial factors such that $T^\mu p^*(p_*(g)) = g$ holds for some Laurent monomial $T^\mu \in \mathbb{C}[T_\varrho^{\pm 1}; \varrho \in \mathbb{R}]$. The *shift* of g is the unique $g' \in \mathbb{C}[T_{\varrho'}; \varrho' \in \mathbb{R}']$ without monomial factors satisfying $p'_*(g') = p_*(g)$.

Definition 2.2. Let $X \subseteq Z_\Sigma$ be the minimal toric embedding of a complete variety given by a complete intersection Cox ring

$$\mathcal{R}(X) = \mathbb{C}[T_\varrho; \varrho \in \mathbb{R}] / \langle g_1, \dots, g_s \rangle.$$

- (i) We call the modification $X' \rightarrow X$ arising from a subdivision $\Sigma' \rightarrow \Sigma$ of fans a *tropical resolution of singularities* if Σ' subdivides $\Sigma \sqcap \text{trop}(X)$ and X' is smooth with complete intersection Cox ring defined by the shifts g'_i of g_i :

$$\mathcal{R}(X') = \mathbb{C}[T_{\varrho'}; \varrho' \in \mathbb{R}'] / \langle g'_1, \dots, g'_s \rangle.$$

- (ii) We say that X is *strongly tropically resolvable* if every subdivision of $\Sigma \sqcap \text{trop}(X)$ admits a regular refinement providing a tropical resolution of singularities.

Assertions (ii) to (v) of Theorem 1.4 will be obtained as a consequence of the following description of discrepancies of a tropical resolution.

Proposition 2.3. *Let $\varphi: X' \rightarrow X$ be a tropical resolution of singularities given by subdivision $\Sigma' \rightarrow \Sigma$ of fans. Then the discrepancy α_ϱ along a divisor D_ϱ corresponding to a ray $\varrho \in \Sigma'$ satisfies*

$$\alpha_\varrho = \frac{\|v_\varrho\|}{\|v'_\varrho\|} - 1 \text{ if } \varrho \not\subseteq A_X^c, \quad \alpha_\varrho \leq -1 \text{ if } \varrho \subseteq A_X^c.$$

We provide two Lemmas used in the proof of Proposition 2.3 and also later. The first one describes the exponents of a shifted polynomial.

Lemma 2.4. *Consider a subdivision of fans $\Sigma' \rightarrow \Sigma$ with Cox constructions given by $P': \mathbb{Z}^{R'} \rightarrow \mathbb{Z}^n$ and $P: \mathbb{Z}^R \rightarrow \mathbb{Z}^n$, a polynomial $g = \sum a_\nu T^\nu \in \mathbb{C}[T_\varrho; \varrho \in R]$ without monomial factors and a linear surjection $F: \mathbb{Q}^{R'} \rightarrow \mathbb{Q}^R$ with $P' = P \circ F$. Then there is a unique $e_F \in \mathbb{Z}^{R'}$ such that the shift g' is given as*

$$g' = T^{e_F} \sum a_\nu T^{F^*(\nu)} \in \mathbb{C}[T_{\varrho'}; \varrho' \in R'].$$

In particular the exponents of g (the vertices of $B(g)$) correspond to the exponents of g' (the vertices of $B(g')$). Moreover, for any exponent ν of g , the corresponding exponent ν' of g' satisfies $\nu'_\varrho = \nu_\varrho$ for all $\varrho \in R \subseteq R'$.

Proof. Choose linear maps $\mu: \mathbb{Z}^{R'} \rightarrow \mathbb{Z}^{R'}$ and $\alpha: \mathbb{Z}^{R'} \rightarrow \mathbb{Z}^R$, both of full rank, such that $F \circ \mu = \alpha$ holds. Then $(p \circ \alpha)^*(p_*(g))$ equals $(p' \circ \mu)^*(p'_*(g'))$ which proves the displayed equality. Choosing an F given by a matrix $[E_r, F']$ with the $r \times r$ unit matrix E_r gives the last statement. \square

For a polynomial g , we denote by $\exp(g)$ the set of its exponent vectors and, as earlier, by $\mathcal{N}(B(g))$ the normal fan of its Newton polytope. Moreover, for a cone $\sigma \in \Sigma$ we denote by $\hat{\sigma} \in \hat{\Sigma}$ the unique cone with $P(\hat{\sigma}) = \sigma$.

Lemma 2.5. *Let $h \in \mathbb{C}[T_1, \dots, T_n]$ and $e \in \mathbb{Z}^R$ such that $g := T^e p^*(h)$ is a polynomial in $\mathbb{C}[T_\varrho; \varrho \in R]$ having no monomial factors. Consider a face $C \subseteq B(h)$, the corresponding cone $\tau \in \mathcal{N}(B(h))$ and suppose that $\sigma \in \Sigma$ satisfies $\text{relint}(\sigma) \subseteq \text{relint}(\tau)$. Then we have*

$$\exp(g) \cap \hat{\sigma}^\perp = P^*(C \cap \exp(h)) + e.$$

Proof. To verify “ \subseteq ”, let $e_g \in \exp(g) \cap \hat{\sigma}^\perp$. Then $e_g = P^*(u_h) + e$ holds with some $u_h \in \exp(h)$. Choose $\hat{v} \in \text{relint}(\hat{\sigma})$. Then, for any $u \in \exp(h)$, we have

$$\langle u, P(\hat{v}) \rangle = \langle P^*(u) + e, \hat{v} \rangle - \langle e, \hat{v} \rangle, \quad \langle P^*(u) + e, \hat{v} \rangle \geq 0.$$

Moreover, $e_g \in \hat{\sigma}^\perp$ implies $\langle P^*(u_h) + e, \hat{v} \rangle = 0$. Thus $u_h \in \exp(h)$ minimizes $P(\hat{v})$. Since $P(\hat{v}) \in \text{relint}(\tau)$ holds, we obtain $u_h \in C$.

For “ \supseteq ”, let $u_h \in C \cap \exp(h)$. Then $e_g := P^*(u_h) + e$ lies in $\exp(g)$. By monomial freeness of g , we find that every ray ϱ of σ admits an $\nu_\varrho \in \exp(g)$ with $\langle \nu_\varrho, e_\varrho \rangle = 0$. Write $\nu_\varrho = P^*(u_\varrho) + e$ with $u_\varrho \in \exp(h)$. Then

$$0 = \langle \nu_\varrho, e_\varrho \rangle = \langle u_\varrho, P(e_\varrho) \rangle + \langle e, P(e_\varrho) \rangle \geq \langle u_h, P(e_\varrho) \rangle + \langle e, P(e_\varrho) \rangle = \langle e_g, e_\varrho \rangle \geq 0$$

holds, where the estimate in the middle is due to the fact that u_h minimizes $P(e_\varrho) \in \tau$. In particular e_g annihilates $\hat{\sigma}$. \square

Proof of Proposition 2.3. We write $R \subseteq \Sigma$ and $R' \subseteq \Sigma'$ for the respective sets of rays. The exceptional divisors of $\varphi: X' \rightarrow X$ are precisely the divisors $D_{X'}^{\varrho'}$, obtained as pullbacks of the toric divisors in Z_Σ given by the rays $\varrho' \in R' \setminus R$; see [4, Prop. 3.14]. We fix such ϱ' and compute the discrepancy of $\varphi: X' \rightarrow X$ along $D_{X'}^{\varrho'}$.

Let $B := B(g_1) + \dots + B(g_s)$ and $B' := B(g'_1) + \dots + B(g'_s)$ be the Minkowski sums of the Newton polytopes $B(g_i)$ and $B(g'_i)$. The inverse image $P^{-1}(\varrho')$ is contained in a maximal cone $\tau \in \mathcal{N}(B(-\mathcal{K}_X) + B)$. Let $\eta \in B(-\mathcal{K}_X) + B$ be the vertex

corresponding to τ . Then $\eta = \nu_{-\mathcal{K}_X} + \nu$ with vertices $\nu_{-\mathcal{K}_X} \in B(-\mathcal{K}_X)$ and $\nu \in B$. Moreover, we write $\nu' \in B'$ for the vertex corresponding to $\nu \in B$ in the sense of Lemma 2.4.

We compute the discrepancy of $\varphi: X' \rightarrow X$ along the divisor $D_{X'}^{\varrho'}$, using the following representatives of the canonical classes of X and X' :

$$D_X^c := \sum_{\varrho \in R} (-1 + \nu_{\varrho}) D_X^{\varrho}, \quad D_{X'}^c := \sum_{\varrho \in R'} (-1 + \nu'_{\varrho}) D_{X'}^{\varrho}.$$

Note that by the definition of a tropical resolution of singularities, $D_{X'}^c$ is indeed a canonical divisor. Moreover, $D_{X'}^c - \varphi^* D_X^c$ is supported on the exceptional locus by Lemma 2.4.

Let $\sigma \in \Sigma$ be the cone with $\text{relint}(\varrho') \subseteq \text{relint}(\sigma)$. Then, on the corresponding chart $X_{\sigma} = X \cap Z_{\sigma}$, the divisor D_X^c is (rationally) principal. More precisely, we claim that on X_{σ} this divisor has a presentation

$$D_X^c = \frac{1}{m} \text{div}(\chi^{mu}) \quad \text{with } u := (P^*)^{-1}(\nu_{-\mathcal{K}_X} + \nu - e_{\Sigma}),$$

where $m \in \mathbb{Z}_{>0}$ is such that mu is integral and χ^{mu} denotes the pullback of the toric character function on Z_{Σ} associated to mu .

To verify the claim, we first show $\langle \nu_{-\mathcal{K}_X}, e_{\varrho} \rangle = 0$ for all rays ϱ of σ . Indeed, due to ampleness of the anticanonical class, $B(-\mathcal{K}_X) \cap \text{relint}(\widehat{\sigma}^{\perp} \cap \gamma_R)$ is non-empty, see [4, Prop. 4.1], and thus contains some element e^* . Because of $\text{relint}(\varrho') \subseteq \text{relint}(\sigma)$, the preimage $P^{-1}(\varrho')$ contains a vector $\mu = \sum_{\varrho \in \sigma(1)} b_{\varrho} e_{\varrho}$ with positive b_{ϱ} . We have $\langle e^*, \mu \rangle = 0$. Since $\nu_{-\mathcal{K}_X} \in B(-\mathcal{K}_X)$ is a minimizing vertex for μ , we conclude $\langle \nu_{-\mathcal{K}_X}, \mu \rangle = 0$ and hence $\langle \nu_{-\mathcal{K}_X}, e_{\varrho} \rangle = 0$ for all rays ϱ of σ . Consequently, on X_{σ} , we have

$$\frac{1}{m} \text{div}(\chi^{mu}) = \sum_{\varrho \in \sigma(1)} \langle u, v_{\varrho} \rangle D_X^{\varrho} = \sum_{\varrho \in \sigma(1)} \langle P^* u, e_{\varrho} \rangle D_X^{\varrho} = \sum_{\varrho \in \sigma(1)} \langle \nu - e_{\Sigma}, e_{\varrho} \rangle D_X^{\varrho}.$$

Using the presentation of D_X^c on X_{σ} just obtained, we see that the discrepancy $a_{\varrho'}$ of $\varphi: X' \rightarrow X$ along $D_{X'}^{\varrho'}$ is the multiplicity of $D_{X'}^c - \text{div}(\chi^u)$ along $D_{X'}^{\varrho'}$, and thus is concretely given by

$$a_{\varrho'} = -1 + \nu'_{\varrho'} - \langle u, v_{\varrho'} \rangle.$$

We show that $\nu'_{\varrho'} = 0$ holds. First note that $\nu = \nu_1 + \dots + \nu_s$, where ν_i is an exponent vector of g_i . Let ν'_i be the corresponding exponent vector of g'_i in the sense of Lemma 2.4. Then $\nu' = \nu'_1 + \dots + \nu'_s$. We claim that $\nu'_{i, \varrho'} = 0$ for all $i = 1, \dots, s$. By definition, ν'_i lies in the face of $B(g'_i)$ which is cut out by $P'^{-1}(\varrho')$. Consequently, the corresponding exponent vector of the pushed down equation $p_*(g_i)$ lies in the face of $B(p_*(g_i))$ that is cut out by ϱ' . Lemma 2.5 applied to ϱ' and g'_i yields that ν'_i is orthogonal to $\tilde{\varrho}'$, i.e., we have $\nu_{i, \varrho'} = 0$.

To conclude the proof we have to evaluate $\langle u, v_{\varrho'} \rangle$. For this, consider the maximal cone $\sigma^{\sharp} \in \mathcal{N}(B_X)$ corresponding to the vertex $u \in B_X$. Then we have $\varrho' \subseteq \sigma^{\sharp}$ and the bounding halfspace

$$H := \{v \in \mathbb{Q}^n; \langle u, v \rangle \geq -1\} \subseteq \mathbb{Q}^n$$

of A_X defined by u satisfies $\sigma^{\sharp} \cap A_X = \sigma^{\sharp} \cap H$. If the ray ϱ' is not contained in A_X , then its leaving point $v'_{\varrho'}$ is the intersection point of ϱ' and ∂H . In this case, we obtain

$$\langle u, v_{\varrho'} \rangle = \frac{\|v_{\varrho'}\|}{\|v'_{\varrho'}\|} \langle u, v'_{\varrho'} \rangle = -\frac{\|v_{\varrho'}\|}{\|v'_{\varrho'}\|}, \quad a_{\varrho'} = -1 + \frac{\|v_{\varrho'}\|}{\|v'_{\varrho'}\|}.$$

If $\varrho' \subseteq A_X$ holds, then ϱ' is contained in H . This means $\langle u, v \rangle \geq -1$ for all $v \in \varrho'$. It follows $\langle u, v_{\varrho'} \rangle \geq 0$ and thus $a_{\varrho'} \leq -1$. \square

Proof of Theorem 1.4, Assertions (ii) to (v). We prove the “if” parts first; recall that for them we only require the existence of one tropical resolution. Let $\varphi: X' \rightarrow X$ be such a resolution, given by a subdivision of fan $\Sigma' \rightarrow \Sigma$. Given a ray $\varrho \in \Sigma'$ not belonging to Σ , we have to show that the discrepancy a_ϱ satisfies the desired bounds. For (ii), let A_X^c be bounded. Then $\varrho \not\subseteq A_X^c$ and Proposition 2.3 gives $a_\varrho > -1$. In assertions (iii) to (v) observe that A_X^c is bounded and thus $\varrho \not\subseteq A_X^c$. In (iii), the intersection point of ϱ and $\partial \varepsilon A_X^c$ is $\varepsilon v'_\varrho$. By assumption, $v_\varrho \notin \varepsilon A_X^c$. Thus thus $\|v_\varrho\| > \varepsilon \|v'_\varrho\|$ and Proposition 2.3 gives $a_\varrho > -1 + \varepsilon$. Similarly, for (iv) and (v), the intersection point of ϱ and ∂A_X^c is v'_ϱ , and Proposition 2.3 gives $a_\varrho \geq 0$ in (iv) and $a_\varrho > 0$ in (v).

We turn to the “only if” parts. Here we required that X is strongly tropically resolvable. For (ii), assume that A_X^c is not bounded. Then A_X^c contains a ray ϱ . Let $X' \rightarrow X$ be a tropical resolution with $\varrho \in \Sigma'$. Then $a_\varrho \leq -1$ by Proposition 2.3, a contradiction. In Assertions (iii) to (v), A_X^c is bounded due to (ii). For (iii), assume that εA_X^c contains an integral point $v \neq 0$ and set $\varrho := \text{cone}(v)$. Let $X' \rightarrow X$ a tropical resolution with $\varrho \in \Sigma'$. Then ϱ and $\varepsilon \partial A_X^c$ intersect at $\varepsilon v'_\varrho$. Because $v_\varrho \in \varepsilon A_X^c$, we have $\|v_\varrho\| \leq \varepsilon \|v'_\varrho\|$ and Proposition 2.3 gives $a_\varrho \leq -1 + \varepsilon$, a contradiction. Similarly, for (iv) and (v), assume that A_X^c contains an (inner) integral point $v \neq 0$ generating a ray ϱ that does not belong to Σ . Let $X' \rightarrow X$ a tropical resolution with $\varrho \in \Sigma'$. Proposition 2.3 implies $a_\varrho < 0$ in case (iv) and $a_\varrho \leq 0$ in case (v), a contradiction. \square

Remark 2.6. The assignment $\eta \mapsto \text{cone}(\eta)$ defines an order-preserving bijection between the anticanonical complex A_X^c and the fan $\Sigma \sqcap \text{trop}(X)$.

We conclude the section with some observations that may be drawn for the intersection of A_X with the tropical lineality space.

Definition 2.7. Let $\text{trop}_0(X) \subseteq \text{trop}(X)$ denote the lineality space of the tropical variety. The *lineality part* of the anticanonical complex is the polyhedral complex $A_{X,0}^c := A_X \sqcap \text{trop}_0(X)$.

Proposition 2.8. Let X be a log terminal Fano variety and let $|A_{X,0}^c|$ denote the support of the lineality part of the anticanonical complex A_X^c .

- (i) $|A_{X,0}^c|$ is a full dimensional polytope in $\text{trop}_0(X)$ having the origin as an interior point.
- (ii) If X is ε -log terminal then the origin is the only lattice point of $\varepsilon |A_{X,0}^c|$.
- (iii) If X is canonical then the origin is the only interior lattice point of $|A_{X,0}^c|$.
- (iv) If X is terminal then the origin is the only lattice point of $|A_{X,0}^c|$.

3. FANO VARIETIES WITH TORUS ACTION OF COMPLEXITY ONE

We take a closer look at Fano varieties with a torus action of complexity one. First we recall the approach to rational varieties with torus action of complexity one provided by [7, 5]. The Cox rings of these varieties are precisely the rings obtained in the following way.

Construction 3.1. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \dots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \dots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. The input data are

- a matrix $A := [a_0, \dots, a_r]$ with pairwise linearly independent column vectors $a_0, \dots, a_r \in \mathbb{C}^2$,
- an integral block matrix P of size $(r + s) \times (n + m)$, the columns of which are pairwise different primitive vectors generating \mathbb{Q}^{r+s} as a cone.

$$P = \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix},$$

where d is an $(s \times n)$ -matrix, d' an $(s \times m)$ -matrix and L an $(r \times n)$ -matrix built from tuples $l_i := (l_{i1}, \dots, l_{in_i}) \in \mathbb{Z}_{\geq 1}^{n_i}$ as follows

$$L = \begin{bmatrix} -l_0 & l_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \dots & l_r \end{bmatrix}.$$

Consider the polynomial ring $\mathbb{C}[T_{ij}, S_k]$ in the variables T_{ij} , where $0 \leq i \leq r$, $1 \leq j \leq n_i$, and S_k , where $1 \leq k \leq m$. For every $0 \leq i \leq r$, define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \dots T_{in_i}^{l_{in_i}}.$$

Denote by \mathfrak{I} the set of all triples $I = (i_1, i_2, i_3)$ with $0 \leq i_1 < i_2 < i_3 \leq r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I := g_{i_1, i_2, i_3} := \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}.$$

Let P^* denote the transpose of P , consider the factor group $K := \mathbb{Z}^{n+m}/\text{im}(P^*)$ and the projection $Q: \mathbb{Z}^{n+m} \rightarrow K$. We define a K -grading on $\mathbb{C}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) := Q(e_{ij}), \quad \deg(S_k) := Q(e_k).$$

Then the trinomials g_I just introduced are K -homogeneous, all of the same degree. In particular, we obtain a K -graded factor ring

$$R(A, P) := \mathbb{C}[T_{ij}, S_k; 0 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m] / \langle g_I; I \in \mathfrak{I} \rangle.$$

Remark 3.2. The K -graded ring $R(A, P)$ of Construction 3.1 is a complete intersection: with $g_i := g_{i, i+1, i+2}$ we have

$$\langle g_I; I \in \mathfrak{I} \rangle = \langle g_0, \dots, g_{r-2} \rangle, \quad \dim(R(A, P)) = n + m - (r - 1).$$

We can always assume that P is *irredundant* in the sense that $l_{i1} + \dots + l_{in_i} \geq 2$ holds for $i = 0, \dots, r$; note that a redundant P allows the elimination of variables in $R(A, P)$.

Remark 3.3. The *anticanonical class* of the K -graded ring $R(A, P)$ from Construction 3.1 is

$$\kappa(A, P) := \sum_{i,j} Q(e_{ij}) + \sum_k Q(e_k) - (r-1) \sum_{j=0}^{n_0} l_{0j} Q(e_{0j}) \in K$$

and the *moving cone* of $R(A, P)$ in $K_{\mathbb{Q}}$ is

$$\text{Mov}(A, P) := \bigcap_{i,j} \text{cone}(Q(e_{uv}, e_t; (u, v) \neq (i, j))) \cap \bigcap_k \text{cone}(Q(e_{uv}, e_t; t \neq k)).$$

The K -graded ring $R(A, P)$ is the Cox ring of a Fano variety if and only if $\kappa(A, P)$ belongs to the relative interior of $\text{Mov}(A, P)$.

Construction 3.4. Consider the K -graded ring $R(A, P)$ of Construction 3.1 and assume that $\kappa(A, P)$ lies in the relative interior of $\text{Mov}(A, P)$. Then the K -grading on $\mathbb{C}[T_{ij}, S_k]$ defines an action of the quasitorus $H := \text{Spec } \mathbb{C}[K]$ on $\bar{Z} := \mathbb{C}^{n+m}$ leaving $\bar{X} := V(g_I; I \in \mathfrak{I}) \subseteq \bar{Z}$ invariant. Consider

$$\hat{Z}_c := \{z \in \bar{Z}; f(z) \neq 0 \text{ for some } f \in \mathbb{C}[T_{ij}, S_k]_{\nu \kappa(A, P)}, \nu \in \mathbb{Z}_{>0}\} \subseteq \bar{Z},$$

the set of H -semistable points with respect to the weight $\kappa(A, P)$. Then $\hat{X} := \bar{X} \cap \hat{Z}_c$ is an open H -invariant set in \bar{X} and we have a commutative diagram

$$\begin{array}{ccc} \hat{X} & \longrightarrow & \hat{Z}_c \\ \parallel H \downarrow & & \downarrow \parallel H \\ X(A, P) & \longrightarrow & Z_c \end{array}$$

where $X(A, P)$ is a Fano variety with torus action of complexity one, $Z_c := \widehat{Z}_c // H$ is a toric Fano variety, the downward maps are characteristic spaces and the lower horizontal arrow is a closed embedding. We have

$$\begin{aligned} \dim(X(A, P)) &= s + 1, & \text{Cl}(X(A, P)) &\cong K, \\ -\mathcal{K}_X &= \kappa(A, P), & \mathcal{R}(X) &\cong R(A, P). \end{aligned}$$

By the results of [7, 5] every normal rational Fano variety with a torus action of complexity one arises from this construction.

Remark 3.5. The following elementary column and row operations on the defining matrix P do not change the isomorphism type of the associated Fano variety $X(A, P)$; we call them *admissible operations*:

- (i) swap two columns inside a block $v_{ij_1}, \dots, v_{ij_{n_i}}$,
- (ii) swap two whole column blocks $v_{ij_1}, \dots, v_{ij_{n_i}}$ and $v_{i'j_1}, \dots, v_{i'j_{n_{i'}}}$,
- (iii) add multiples of the upper r rows to one of the last s rows,
- (iv) any elementary row operation among the last s rows,
- (v) swap two columns inside the d' block.

The operations of type (iii) and (iv) do not change the associated ring $R(A, P)$, whereas the types (i), (ii), (v) correspond to certain renumberings of the variables of $R(A, P)$ keeping the (graded) isomorphism type.

We now discuss the resolution of singularities in this setting. The references for complete proofs are [1, Sec. 3.4.4] and [8]. A local version of our desingularization using another approach was given in [14].

Construction 3.6. Consider the setting of Construction 3.4. Let $\gamma \subseteq \mathbb{Q}^{n+m}$ be the positive orthant and for a face $\gamma_0 \preceq \gamma$ let $\gamma_0^* := \gamma_0^\perp \cap \gamma$ be the complementary face. Then the fan of $Z_c = \widehat{Z}_c // H$ is

$$\Sigma_c = \{P(\gamma_0^*); \gamma_0 \preceq \gamma, \kappa(A, P) \in \text{relint}(Q(\gamma_0))\}.$$

In particular, the primitive generators of the rays of Σ are precisely the columns v_{ij} and v_k of the matrix P . With $P_0 = [L, 0]$ and $P_1 = [E_r, 0]$, where E_r is the $r \times r$ unit matrix, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^{n+m} & \xrightarrow{P} & \mathbb{Z}^{r+s} \\ & \searrow P_0 \quad \swarrow P_1 & \\ & \mathbb{Z}^r & \end{array}$$

The torus T acting on $X(A, P)$ is the subtorus $T \subseteq \mathbb{T}^{r+s}$ corresponding to the sublattice $\ker(P_1) = 0 \times \mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$. Now, let $e_1, \dots, e_r \in \mathbb{Z}^r$ be the canonical basis vectors, set

$$\varrho_0 := \text{cone}(-e_1 - \dots - e_r), \quad \varrho_i := \text{cone}(e_i), \quad 1 \leq i \leq r,$$

and consider the fan $\Delta(r) := \{0, \varrho_0, \dots, \varrho_r\}$ in \mathbb{Z}^r . Note that P_1 sends an ij -th column v_{ij} of P into the ray ϱ_i and the columns v_k to zero. The tropical variety of $X \cap \mathbb{T}^n \subseteq Z_c$ is then given as

$$\text{trop}(X) = \bigcup_{i=0}^r P_1^{-1}(\varrho_i) \subseteq \mathbb{Q}^{r+s}.$$

The minimal toric ambient variety $Z_\Sigma \subseteq Z_c$ of $X \subseteq Z_c$ is the open toric subvariety having as closed orbits the minimal orbits of Z_c intersecting X . The fan Σ of Z_Σ is generated by the cones of Σ_c with $\text{relint}(\sigma) \cap \text{trop}(X) \neq \emptyset$. Set

$$\Sigma' := \Sigma \sqcap \text{trop}(X) = \{\sigma \cap P_1^{-1}(\varrho_i); \sigma \in \Sigma, 0 \leq i \leq r\}.$$

Then we have a map of fans $\Sigma' \rightarrow \Sigma$ and the associated birational toric morphism $Z_{\Sigma'} \rightarrow Z_{\Sigma}$ fits into a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & Z_{\Sigma'} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z_{\Sigma} \end{array}$$

where $X' \subseteq Z_{\Sigma'}$ is the proper transform, i.e., the closure of $X \cap \mathbb{T}^{r+s}$ in $Z_{\Sigma'}$. Any regular subdivision $\Sigma'' \rightarrow \Sigma'$ provides a toric resolution $Z_{\Sigma''} \rightarrow Z_{\Sigma'}$ and induces a resolution $X'' \rightarrow X'$.

The resulting varieties X' and X'' arising in this construction are again normal rational varieties with torus action of complexity one and have Cox rings of the form $R(A, P')$ and $R(A, P'')$ as presented in Construction 3.1, see [1, Thm. 3.4.4.9]. In particular they are Mori dream spaces and we obtain the following.

Proposition 3.7. *Let $X = X(A, P)$ be a Fano variety as in Construction 3.4. Then X is strongly tropically resolvable.*

4. STRUCTURE OF THE ANTICANONICAL COMPLEX

The notation is the same as in Section 3. We consider a \mathbb{Q} -factorial rational Fano variety $X = X(A, P)$ with torus action of complexity one and investigate the structure of the associated anticanonical complex A_X^c . Combining the results with Theorem 1.4, we derive first bounding conditions on the entries of the defining matrix P .

Recall that we have $X \subseteq Z_{\Sigma} \subseteq Z_c$, where Z_c is a toric Fano variety and Z_{Σ} is the minimal open toric subvariety of Z_c containing X as a closed subvariety. The fans Σ_c of Z_c and Σ of Z_{Σ} live in the lattice \mathbb{Z}^{r+s} . They share the same set of rays ϱ and the primitive generators $v_{\varrho} \in \varrho$ are precisely the columns of the matrix

$$P = \begin{pmatrix} -l_0 & l_1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -l_0 & 0 & & l_r & 0 \\ d_0 & d_1 & & d_r & d' \end{pmatrix},$$

where each $l_i = (l_{i1}, \dots, l_{in_i})$ is a $1 \times n_i$ block and each $d_i = (d_{i1}, \dots, d_{in_i})$ is an $s \times n_i$ block. The tropical variety $\text{trop}(X)$ with its quasifan structure also lives in \mathbb{Z}^{r+s} . With $\lambda := 0 \times \mathbb{Q}^s \subseteq \mathbb{Q}^{r+s}$, the canonical basis vectors e_1, \dots, e_r and $e_0 := -e_1 - \dots - e_r$, we have

$$\text{trop}(X) = \tau_0 \cup \dots \cup \tau_r \subseteq \mathbb{Q}^{r+s}, \quad \text{where } \tau_i := \text{cone}(e_i) + \lambda.$$

Note that this defines the coarsest possible quasifan structure on $\text{trop}(X)$, and the lineality space of this quasifan is λ .

Definition 4.1. A cone $\sigma \in \Sigma$ is called *big*, if $\sigma \cap \text{relint}(\tau_i) \neq \emptyset$ holds for each $i = 0, \dots, r$. An *elementary big* cone is a big cone $\sigma \in \Sigma$ having no rays inside λ and precisely one inside τ_i for each $i = 0, \dots, r$. A *leaf cone* is a $\sigma \in \Sigma$ such that $\sigma \subseteq \tau_i$ holds for some i .

Remark 4.2. The big cones and the leaf cones are precisely those cones $\sigma \in \Sigma$ such that $\text{relint}(\sigma)$ intersects $\text{trop}(X)$. The latter property, by Tevelev's criterion [16, Lemma 2.2], merely means that the big cones and the leaf cones describe precisely the toric orbits of Z intersecting X . Observe that all maximal cones of Σ are big cones or leaf cones.

Definition 4.3. Let $\sigma \in \Sigma$ be an elementary big cone. We assign the following positive integers to the rays $\varrho = \text{cone}(v_{ij}) \in \sigma^{(1)}$ of σ and to σ itself:

$$l_\varrho := l_{ij}, \quad \ell_{\sigma, \varrho} := l_\sigma^{-1} \prod_{\varrho' \in \sigma^{(1)}} l_{\varrho'}, \quad \ell_\sigma := \sum_{\varrho \in \sigma^{(1)}} \ell_{\sigma, \varrho} - (r-1) \prod_{\varrho \in \sigma^{(1)}} l_\varrho.$$

Moreover, in \mathbb{Q}^{r+s} , we define vectors and a ray:

$$v_\sigma := \sum_{\varrho \in \sigma^{(1)}} \ell_{\sigma, \varrho} v_\varrho, \quad v'_\sigma := \ell_\sigma^{-1} v_\sigma, \quad \varrho_\sigma := \text{cone}(v_\sigma).$$

Finally, we denote by c_σ the greatest common divisor of the entries of the vector $v_\sigma \in \mathbb{Z}^{r+s}$.

The first structural statement describes the rays of the coarsest common refinement $\Sigma \sqcap \text{trop}(X)$ of the fan Σ and the tropical variety $\text{trop}(X)$ regarded as a quasifan.

Proposition 4.4. *Let $X = X(A, P)$ be a \mathbb{Q} -factorial Fano variety.*

- (i) *For every elementary big cone $\sigma \in \Sigma$, we have $\sigma \cap \lambda = \varrho_\sigma$; in particular, ϱ_σ lies in the lineality space λ .*
- (ii) *The set of rays of $\Sigma \sqcap \text{trop}(X)$ consists of the rays $\varrho \in \Sigma$ and the rays ϱ_σ , where $\sigma \in \Sigma$ runs through the elementary big cones.*

Proof. For (i), one directly computes the intersection $\sigma \cap \lambda$. We prove (ii). Since all rays of Σ lie on $\text{trop}(X)$, the rays of Σ are also rays of $\Sigma \sqcap \text{trop}(X)$. By (i), the ϱ_σ , where $\sigma \in \Sigma$ is elementary big, are rays of $\Sigma \sqcap \text{trop}(X)$. Let $\varrho' \in \Sigma \sqcap \text{trop}(X)$ be any ray not belonging to Σ . Then there exist cones $\sigma \in \Sigma$ and $\tau \in \text{trop}(X)$ which satisfy $\sigma \cap \tau = \varrho'$ and which are minimal with this property. The latter means that $\text{relint}(\varrho') = \text{relint}(\sigma) \cap \text{relint}(\tau)$.

To obtain $\tau = \lambda$, we have to exclude the case $\tau = \tau_i$ for some $i = 0, \dots, r$. Indeed if $\tau = \tau_i$, then no ray $\varrho \preceq \sigma$ lies in τ_i , because otherwise we had $\varrho \subseteq \sigma \cap \tau_i = \varrho'$, contradicting $\varrho' \notin \Sigma$. Thus, σ has no rays inside τ_i . Since all rays of σ lie on $\text{trop}(X)$, we conclude $\text{relint}(\sigma) \cap \text{relint}(\tau_i) = \emptyset$, a contradiction.

We show that σ is an elementary big cone. Firstly, σ must be big because otherwise we had $\text{relint}(\sigma) \cap \lambda = \emptyset$. Since X is \mathbb{Q} -factorial, σ is simplicial. Thus there exists an elementary big face η of σ . But then $\varrho_\eta = \eta \cap \lambda \preceq \sigma \cap \lambda = \varrho'$ which implies $\varrho' = \varrho_\eta$. By minimality of σ , we conclude $\sigma = \eta$. \square

We take a closer look at the discrepancies of a tropical resolution of singularities along the divisors corresponding to the rays ϱ_σ .

Proposition 4.5. *Let $X = X(A, P)$ be a \mathbb{Q} -factorial Fano variety and $\sigma \in \Sigma$ an elementary big cone.*

- (i) *If ϱ_σ leaves A_X , e.g. if σ defines a log terminal singularity, then its leaving point is $v'_{\varrho_\sigma} = \ell_\sigma^{-1} v_\sigma = v'_\sigma$.*
- (ii) *For any tropical resolution $\varphi: X' \rightarrow X$ of singularities, the discrepancy along the divisor corresponding to ϱ_σ is $a_{\varrho_\sigma} = -1 + c_\sigma^{-1} \ell_\sigma$.*

Proof. Recall that the intersection point v'_{ϱ_σ} of the ray ϱ_σ with the boundary ∂A_X^c is defined by

$$\langle u, v'_{\varrho_\sigma} \rangle = -1, \quad \text{where } u := (P^*)^{-1}(e_{-\mathcal{K}_X} + e - e_\Sigma)$$

with any vertex $e_{-\mathcal{K}_X} + e - e_\Sigma$ of $B(-\mathcal{K}_X) + B - e_\Sigma$ minimizing $\widehat{v}_\sigma := \sum_{\varrho \in \sigma^{(1)}} \ell_{\sigma, \varrho} e_\varrho$. For $v_\sigma = P(\widehat{v}_\sigma)$, we obtain

$$\langle u, v_\sigma \rangle = \langle e_{-\mathcal{K}_X}, \widehat{v}_\sigma \rangle + \langle e, \widehat{v}_\sigma \rangle - \langle e_\Sigma, \widehat{v}_\sigma \rangle = \langle e, \widehat{v}_\sigma \rangle - \langle e_\Sigma, \widehat{v}_\sigma \rangle.$$

To compute further, set $\hat{u}_i := \sum_{\varrho \in R_i} l_\varrho e_\varrho$ for $i = 0, \dots, r$, where R_i denotes the set of rays of Σ contained in τ_i . Denoting by ϱ_i the unique ray of σ in τ_i , we have

$$\langle \hat{u}_i, \hat{v}_\sigma \rangle = l_{\varrho_i} \ell_{\sigma, \varrho_i} = \prod_{\varrho \in \sigma^{(1)}} l_\varrho.$$

Consequently, for any point $e \in B = B(g_0) + \dots + B(g_{r-2})$, we obtain

$$\langle e, \hat{v}_\sigma \rangle = (r-1) \prod_{\varrho \in \sigma^{(1)}} l_\varrho.$$

Thus, we obtain $\langle u, v_\sigma \rangle = -\ell_\sigma$ and the leaving point is $v'_{\varrho_\sigma} = \ell_\sigma^{-1} v_\sigma = v'_\sigma$ as claimed in (i). Assertion (ii) is then a direct application of Proposition 2.3. \square

As an application, we obtain first bounding conditions on the entries l_ϱ of the defining matrix P in terms of the singularities of X .

Corollary 4.6. *Let $X = X(A, P)$ be a \mathbb{Q} -factorial Fano variety and $\sigma \in \Sigma$ an elementary big cone. If the singularity defined by σ is*

- (i) *log terminal, then $\sum_{\varrho \in \sigma^{(1)}} l_\varrho^{-1} > r-1$,*
- (ii) *ε -log terminal, then $\sum_{\varrho \in \sigma^{(1)}} l_\varrho^{-1} > r-1 + \varepsilon c_\sigma \prod_{\varrho \in \sigma^{(1)}} l_\varrho^{-1}$,*
- (iii) *canonical, then $\sum_{\varrho \in \sigma^{(1)}} l_\varrho^{-1} \geq r-1 + c_\sigma \prod_{\varrho \in \sigma^{(1)}} l_\varrho^{-1}$,*
- (iv) *terminal, then $\sum_{\varrho \in \sigma^{(1)}} l_\varrho^{-1} > r-1 + c_\sigma \prod_{\varrho \in \sigma^{(1)}} l_\varrho^{-1}$.*

Corollary 4.7. *Let $X = X(A, P)$ be a \mathbb{Q} -factorial Fano variety and consider an elementary big cone $\sigma = \varrho_0 + \dots + \varrho_r \in \Sigma$ defining a log terminal singularity. Assume $l_{\varrho_0} \geq \dots \geq l_{\varrho_r}$. Then $l_{\varrho_3} = \dots = l_{\varrho_r} = 1$ holds and $(l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2})$ is a platonic triple, i.e., one of*

$$(l_{\varrho_0}, l_{\varrho_1}, 1), \quad (l_{\varrho_0}, 2, 2), \quad (3, 3, 2), \quad (4, 3, 2), \quad (5, 3, 2).$$

According to these possibilities, the number ℓ_σ is given as

$$\begin{aligned} \ell_\sigma &= l_{\varrho_0} l_{\varrho_1} + l_{\varrho_0} l_{\varrho_2} + l_{\varrho_1} l_{\varrho_2} - l_{\varrho_0} l_{\varrho_1} l_{\varrho_2} \\ &= \begin{cases} l_{\varrho_0} + l_{\varrho_1}, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (l_{\varrho_0}, l_{\varrho_1}, 1), \\ 4, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (l_{\varrho_0}, 2, 2), \\ 3, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (3, 3, 2), \\ 2, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (4, 3, 2), \\ 1, & \text{if } (l_{\varrho_0}, l_{\varrho_1}, l_{\varrho_2}) = (5, 3, 2). \end{cases} \end{aligned}$$

Corollary 4.8. *Let $X = X(A, P)$ be a log terminal \mathbb{Q} -factorial Fano variety. Assume that P is irredundant and Σ contains a big cone. Then the number $r-1$ of relations is bounded by*

$$r-1 \leq \dim(X) + \text{rk}(\text{Pic}(X)).$$

Proof. Since X is \mathbb{Q} -factorial, $\text{Pic}(X)$ is of rank $n + m - r - s$. Let $I \subseteq \{0, \dots, r\}$ be the set of indices with $n_i > 1$ and set $n_I := \sum_{i \in I} n_i$. Then the rank of $\text{Pic}(X)$ equals $n_I + m - |I| - s$. Since there exists a big cone, there is also an elementary big cone $\sigma = \varrho_0 + \dots + \varrho_r \in \Sigma$. Since P is irredundant, $l_{\varrho_i} > 1$ holds for all $i \notin I$. Corollary 4.7 yields $|I| \geq r-2$. We conclude

$$\text{rk}(\text{Pic}(X)) = m + n_I - |I| - s \geq 2|I| - |I| - s \geq r-2-s = r-1-\dim(X). \quad \square$$

Definition 4.9. Let A_X^c be the anticanonical complex of $X = X(A, P)$. Recall that the lineality part of A_X^c is the polyhedral complex $A_{X,0}^c = A_X^c \sqcap \lambda$. The i -th leaf of A_X^c is the polyhedral complex $A_X^c \sqcap \tau_i$.

Corollary 4.10. *Let $X = X(A, P)$ be a log terminal \mathbb{Q} -factorial Fano variety. Then the vertices of the anticanonical complex A_X^c are precisely the points v_ϱ and v'_σ , where ϱ runs through the rays and σ through the elementary big cones of Σ . In particular, for the supports of the lineality part and the leaves of A_X^c , we obtain*

$$\begin{aligned} |A_X^c \cap \lambda| &= \text{conv}(v_\varrho, v'_\sigma; \varrho \in \Sigma \text{ with } \varrho \subseteq \lambda, \sigma \in \Sigma \text{ elementary big}), \\ |A_X^c \cap \tau_i| &= \text{conv}(v_\varrho, v'_\sigma; \varrho \in \Sigma \text{ with } \varrho \subseteq \tau_i, \sigma \in \Sigma \text{ elementary big}). \end{aligned}$$

Remark 4.11. Let $X = X(A, P)$ be a \mathbb{Q} -factorial Fano variety and X' the variety arising from the tropical refinement $\Sigma \sqcap \text{trop}(X)$. Then $A_{X'}^c$ and A_X^c both generate $\Sigma \sqcap \text{trop}(X)$ but do not in general coincide, because the rays ϱ_σ of big elementary cones $\sigma \in \Sigma$ intersect the boundary of $A_{X'}^c$ in integral points, whereas the intersection points v'_σ with A_X^c do not need to be integral.

Remark 4.12. The anticanonical complex A_X^c of a Fano variety $X = X(A, P)$ can also be obtained in the following way. Since the defining relations g_l of $\mathcal{R}(X)$ all have the same K -degree, we may define A_X^c in a slightly different way by exchanging $B(g)$ for

$$B(g)' := (r-1)\text{conv}(\mu_0, \dots, \mu_r),$$

where μ_0, \dots, μ_r are the exponent vectors occurring in g_0, \dots, g_{r-2} . Then $(r-1)u_l - e_\Sigma$ is a representative of $-\mathcal{K}_X$ and all the proofs work in exactly the same way. On the pro side we note that $B(g)'$ does not depend on the enumeration of the variables T_ϱ , while $B(g)$ does.

5. TERMINAL FANO THREEFOLDS

Here we show how to obtain the classification of terminal \mathbb{Q} -factorial Fano threefolds X of Picard number one coming with an effective action of a two-dimensional torus given in Theorem 1.1. First recall the following.

Remark 5.1. For any Fano variety X with at most log terminal singularities, the divisor class group $\text{Cl}(X)$ is finitely generated; see [11, Sec. 2.1]. If X comes in addition with a torus action of complexity one, then X is rational and its Cox ring is finitely generated; see [1, Remark IV.4.1.5].

This allows us to work in terms of the defining data (A, P) of X and the notation of Constructions 3.1 and 3.4, where we always choose P to be irredundant. The main step is to derive suitable effective bounds on the entries of P . According to Theorem 1.4, terminality of X is equivalent to the fact that the anticanonical complex A_X^c contains no lattice points except the origin and the columns of the defining matrix P . A first observation towards bounds for the shape of P is that log-terminality leads to the following situations.

Lemma 5.2. *Let $X = X(A, P)$ a non-toric log terminal \mathbb{Q} -factorial Fano threefold of Picard number one, where P is irredundant. Then, after suitable admissible operations, P suits into one of the following cases:*

- (i) $m = 0$, $r = 2$ and $n = 5$, where $n_0 = n_1 = 2$, $n_2 = 1$.
- (ii) $m = 0$, $r = 3$ and $n = 6$, where $n_0 = n_1 = 2$, $n_2 = n_3 = 1$.
- (iii) $m = 0$, $r = 4$ and $n = 7$, where $n_0 = n_1 = 2$, $n_2 = n_3 = n_4 = 1$.
- (iv) $m = 0$, $r = 2$ and $n = 5$, where $n_0 = 3$, $n_1 = n_2 = 1$.
- (v) $m = 0$, $r = 3$ and $n = 6$, where $n_0 = 3$, $n_1 = n_2 = n_3 = 1$.
- (vi) $m = 1$, $r = 2$ and $n = 4$, where $n_0 = 2$, $n_1 = n_2 = 1$.
- (vii) $m = 1$, $r = 3$ and $n = 5$, where $n_0 = 2$, $n_1 = n_2 = n_3 = 1$.
- (viii) $m = 2$, $r = 2$ and $n = 3$, where $n_0 = n_1 = n_2 = 1$.

Proof. Since X is non-toric, there is at least one relation in the Cox ring. This implies $r \geq 2$. Since X is of Picard number one, there is an elementary big cone and thus Corollary 4.8 yields $r \leq 5$. Using $n + m = \dim(X) + r$, we obtain

$$2 \leq r \leq 5, \quad r + 1 \leq n, \quad n + m = r + 3.$$

Combining Corollary 4.7 with the fact that P is irredundant, we see that at most three of the n_i equal one. This leaves us with the cases listed in the assertion. \square

We treat exemplarily Situation (i) of Lemma 5.2. This case reflects all the occurring arguments. The final bounds on the defining matrix P are given in Propositions 5.16 to 5.18.

Proposition 5.3. *Let $X = X(A, P)$ a non-toric terminal \mathbb{Q} -factorial Fano threefold of Picard number one such that P is irredundant and we have $r = 2$, $m = 0$ and $n = 5$, where $n_0 = n_1 = 2$, $n_2 = 1$. Then $l_{01} = l_{02} = 1$ or $l_{11} = l_{12} = 1$ hold.*

Proof. Since P is irredundant, we have $l_{21} \geq 2$. Moreover, by suitable admissible operations, we achieve $l_{01} \geq l_{11} \geq l_{12}$, $l_{01} \geq l_{02}$. In total, P is of the form

$$P = \begin{bmatrix} -l_{01} & -l_{02} & l_{11} & l_{12} & 0 \\ -l_{01} & -l_{02} & 0 & 0 & l_{21} \\ d_{101} & d_{102} & d_{111} & d_{112} & d_{121} \\ d_{201} & d_{202} & d_{211} & d_{212} & d_{221} \end{bmatrix}.$$

We have to show that in the case $l_{11} > 1$, no terminal $X = X(A, P)$ is left. According to Corollary 4.7, this means to treat the following configurations of the l_{ij} :

(l_{01}, l_{11}, l_{21})	l_{02}	l_{12}
$(l_{01}, 2, 2)$	$\leq l_{01}$	≤ 2
$(2, 2, l_{21})$	≤ 2	≤ 2
$(3, 3, 2)$	≤ 3	≤ 3
$(3, 2, 3)$	≤ 3	≤ 2
$(4, 3, 2)$	≤ 4	≤ 3
$(4, 2, 3)$	≤ 4	≤ 2
$(3, 2, 4)$	≤ 3	≤ 2
$(5, 3, 2)$	≤ 5	≤ 3
$(5, 2, 3)$	≤ 5	≤ 2
$(3, 2, 5)$	≤ 3	≤ 2

We first consider the linearity part $A_{X,0}^c$ of the anticanonical complex A_X^c . Corollary 4.10 allows an explicit computation. For the vertex u of $A_{X,0}^c$ defined by the cone σ corresponding to the platonic triples from the left column of the table above we obtain coordinates

$$u = \left(0, 0, \frac{l_{01}l_{11}d_{121} + l_{01}l_{21}d_{111} + l_{11}l_{21}d_{101}}{l_{01}l_{11} + l_{01}l_{21} + l_{11}l_{21} - l_{01}l_{11}l_{21}}, \frac{l_{01}l_{11}d_{221} + l_{01}l_{21}d_{211} + l_{11}l_{21}d_{201}}{l_{01}l_{11} + l_{01}l_{21} + l_{11}l_{21} - l_{01}l_{11}l_{21}} \right).$$

The (common) denominator of these coordinates is the ℓ_σ from Corollary 4.7. For triples of type $(5, 3, 2)$ we have $\ell_\sigma = 1$ and thus u is integral. For triples of type $(4, 3, 2)$ we have $\ell_\sigma = 2$ and the numerators are even, because every summand is a multiple of 2 or 4. Thus, u is integral again. Similarly, for triples of type $(3, 3, 2)$, we have $\ell_\sigma = 3$, the numerators are multiples of 3 and u is integral. By Theorem 1.4 this contradicts the terminality of X and we are left with the configurations

l_{01}	l_{02}	l_{11}	l_{12}	l_{21}
2	1	2	1	l_{21}
2	1	2	2	l_{21}
2	2	2	2	l_{21}
l_{01}	l_{02}	2	1	2
l_{01}	l_{02}	2	2	2

In each of the cases, we detect a lattice point on an edge of A_X^c located in a leaf, contradicting again terminality. The procedure is the same for all configurations; we treat exemplarily the first one. There, after suitable admissible operations, the matrix P is of the form

$$P = \begin{bmatrix} -2 & -1 & 2 & 1 & 0 \\ -2 & -1 & 0 & 0 & l_{21} \\ 1 & 0 & d_{111} & 0 & d_{121} \\ 0 & 0 & d_{211} & 0 & d_{221} \end{bmatrix}.$$

According to Corollary 4.10, the vertices of the support of the lineality part $A_{X,0}^c$ of the anticanonical complex A_X^c are given by

$$\begin{aligned} u_1 &:= \left(0, 0, \frac{l_{21}}{2} + \frac{l_{21}}{2}d_{111} + d_{121}, \frac{l_{21}}{2}d_{211} + d_{221} \right), \\ u_2 &:= \left(0, 0, \frac{l_{21}d_{111} + 2d_{121}}{l_{21} + 2}, \frac{l_{21}d_{211} + 2d_{221}}{l_{21} + 2} \right), \\ u_3 &:= \left(0, 0, \frac{l_{21} + 2d_{121}}{l_{21} + 2}, \frac{2d_{221}}{l_{21} + 2} \right), \\ u_4 &:= \left(0, 0, \frac{d_{121}}{l_{21} + 1}, \frac{d_{221}}{l_{21} + 1} \right). \end{aligned}$$

Note that l_{21} is odd, because otherwise u_1 would be a lattice point. Using once again Corollary 4.10, we obtain the following explicit description of the second leaf:

$$|A_X^c \sqcap \tau_2| = \text{conv}(v_{21}, u_1, u_2, u_3, u_4),$$

where v_{21} denotes the last column of the matrix P . Using the fact that l_{21} is odd, we see that on the edge connecting v_{21} to u_1 lies at least one lattice point, namely

$$\frac{l_{21} - 1}{l_{21}}u_1 + \frac{1}{l_{21}}v_{21} = \left(0, 1, d_{121} + (d_{111} + 1)\frac{l_{21} - 1}{2}, d_{221} + d_{211}\frac{l_{21} - 1}{2} \right).$$

Similarly, we find in the remaining cases such a point on an edge of A_X^c connecting a half-integral vertex of $A_{X,0}^c$ with a column of P containing one of the not yet fixed l_{ij} . \square

As a consequence of Proposition 5.3, we can focus our search for terminal varieties $X(A, P)$ on defining matrices P of the following type.

Setting 5.4. Let $X = X(A, P)$ be a non-toric terminal \mathbb{Q} -factorial Fano threefold of Picard number one, such that P is irredundant with $r = 2$, $m = 0$ and $n = 5$, where $n_0 = n_1 = 2$, $n_2 = 1$. Assume that $l_{01} = l_{02} = 1$ holds. Then, after suitable admissible operations, P is of the form

$$P = \begin{bmatrix} -1 & -1 & l_{11} & l_{12} & 0 \\ -1 & -1 & 0 & 0 & l_{21} \\ 0 & 1 & d_{111} & d_{112} & d_{121} \\ 0 & 0 & d_{211} & d_{212} & d_{221} \end{bmatrix},$$

where $l_{11} \geq l_{12}$ and $l_{21} \geq 2$ hold. Moreover, denoting by P_{ij} the matrix obtained by removing the column v_{ij} from P , we have positive *weights*

$$w_{01} := \det(P_{01}), \quad w_{02} := -\det(P_{02}),$$

$$w_{11} := \det(P_{11}), \quad w_{12} := -\det(P_{12}), \quad w_{21} := \det(P_{21}).$$

Observe that the weight vector $(w_{01}, w_{02}, w_{11}, w_{12}, w_{21})$ lies in the kernel of P . The last three weights are explicitly given by

$$w_{11} = -l_{21}d_{212} - l_{12}d_{221}, \quad w_{12} = l_{21}d_{211} + l_{11}d_{221}, \quad w_{21} = -l_{11}d_{212} + l_{12}d_{211}$$

and the first two weights can be expressed in a compact form in terms of the others as follows:

$$w_{02} = -d_{111}w_{11} - d_{112}w_{12} - d_{121}w_{21}, \quad w_{01} = l_{21}w_{21} - w_{02}.$$

Remark 5.5. In Setting 5.4, we can achieve by further admissible operations without changing the shape of P the following for the entries of the third and fourth row of P :

$$0 \leq d_{121}, d_{221} < l_{21}, \quad d_{121} < d_{221} \text{ if } d_{221} \neq 0, \quad 0 \leq d_{112} < w_{11},$$

$$-\frac{(l_{21} + d_{121})w_{21} + d_{112}w_{12}}{w_{11}} < d_{111} < -\frac{d_{121}w_{21} + d_{112}w_{12}}{w_{11}}.$$

For the third estimate we add a suitable multiple of $d_{221}(p_1 - p_2) + l_{21}p_4$ to p_3 , where p_i denotes the i -th row of P (this preserves the first two estimates). The inequalities for d_{111} follow directly from $w_{02} > 0$ and $w_{01} > 0$.

A first series of bounds on the entries of the defining matrix P is derived from the fact that, by terminality, the lineality part $A_{X,0}^c$ of the anticanonical complex A_X^c has the origin as its only lattice point; we also write $A_{X,0}^c$ for the support of the lineality part, which in our situation is a rational two-dimensional polytope. Here is how it precisely looks.

Lemma 5.6. *Let $X = X(A, P)$ be as in Setting 5.4. The vertices of $A_{X,0}^c$ regarded as a subset of the lineality space \mathbb{Q}^2 of the tropical variety are*

$$\begin{aligned} u_1 &:= \left[\frac{l_{21}d_{111} + l_{11}d_{121}}{l_{21} + l_{11}}, \frac{l_{21}d_{211} + l_{11}d_{221}}{l_{21} + l_{11}} \right], \\ u_2 &:= \left[\frac{l_{11}l_{21} + l_{21}d_{111} + l_{11}d_{121}}{l_{21} + l_{11}}, \frac{l_{21}d_{211} + l_{11}d_{221}}{l_{21} + l_{11}} \right], \\ u_3 &:= \left[\frac{l_{21}d_{112} + l_{12}d_{121}}{l_{21} + l_{12}}, \frac{l_{21}d_{212} + l_{12}d_{221}}{l_{21} + l_{12}} \right], \\ u_4 &:= \left[\frac{l_{12}l_{21} + l_{21}d_{112} + l_{12}d_{121}}{l_{21} + l_{12}}, \frac{l_{21}d_{212} + l_{12}d_{221}}{l_{21} + l_{12}} \right]. \end{aligned}$$

Proof. We just compute the lineality part $A_{X,0}^c$ of the anticanonical complex A_X^c according to Corollary 4.10. \square

Remark 5.7. Observe that $A_{X,0}^c$ as described in Lemma 5.6 is a trapezoid. The edges $g_1 := \overline{u_1u_2}$ and $g_2 := \overline{u_3u_4}$ are parallel to the x -axis and the remaining two edges are $\overline{u_1u_3}$ and $\overline{u_2u_4}$. Length and y -value $h(g_i)$ of the line segments g_i are

$$|g_1| = \frac{l_{11}l_{21}}{l_{11} + l_{21}}, \quad h(g_1) = \frac{w_{12}}{l_{11} + l_{21}}, \quad |g_2| = \frac{l_{12}l_{21}}{l_{12} + l_{21}}, \quad h(g_2) = -\frac{w_{11}}{l_{12} + l_{21}}.$$

Since we assume $l_{11} \geq l_{12}$ in Setting 5.4, the lower segment g_2 is shorter than the upper segment g_1 . Note that the values $|g_i|$ and $h(g_i)$ are invariant under admissible row operations of type 3.5 (iii).

Lemma 5.8. *Let $X = X(A, P)$ be as in Setting 5.4. Let $h := h(g_1) - h(g_2)$ denote the total height of the trapezoid $A_{X,0}^c$. Then we have*

$$\frac{l_{12}l_{21}}{l_{12} + l_{21}} < 2, \quad \frac{l_{11}l_{21}}{l_{11} + l_{21}} < \frac{2(l_{12} + l_{21}) - l_{12}l_{21}}{w_{11}} \cdot h + \frac{l_{12}l_{21}}{l_{12} + l_{21}}.$$

Moreover, one has the following estimates

$$w_{01} < w_{11} + w_{12} + w_{21}, \quad w_{02} < w_{11} + w_{12} + w_{21}.$$

Proof. For the first inequality, note that the lower bounding segment g_2 of $A_{X,0}^c$ is of length at most 2, because otherwise the segment $A_{X,0}^c \cap \{y = 0\}$ is of length at least 2 as well, which would imply existence of lattice points different from the origin in A_X^c . Similarly, since $A_{X,0}^c \cap \{y = 0\}$ has length strictly smaller than 2, we arrive at the second inequality:

$$|g_1| < \frac{2 - |g_2|}{|h(g_2)|} \cdot h + |g_2|.$$

Explicitly computing $A_{X,0}^c \cap \{y = 0\}$ gives the bounding x -values $-w_{01}/(w_{11} + w_{12} + w_{21})$ and $w_{02}/(w_{11} + w_{12} + w_{21})$. Since the origin is the only lattice point in $A_{X,0}^c \cap \{y = 0\}$, we arrive at estimates number three and four. \square

Lemma 5.9. *Let $X = X(A, P)$ be as in Setting 5.4. If $l_{21} \geq 3$ holds, then we obtain the estimate*

$$l_{12} < \frac{l_{21} + 2}{l_{21} - 2} \leq 5.$$

Proof. Estimates three and four from Lemma 5.8 imply

$$l_{21}w_{21} = l_{11}w_{11} + l_{12}w_{12} = w_{01} + w_{02} < 2w_{11} + 2w_{12} + 2w_{21}.$$

We deduce

$$(l_{11} - 2)w_{11} + (l_{12} - 2)w_{12} < 2w_{21}, \quad (l_{21} - 2)w_{21} < 2w_{11} + 2w_{12}.$$

Using $l_{21} \geq 3$ we obtain

$$(l_{11} - 2)w_{11} + (l_{12} - 2)w_{12} < \frac{4}{l_{21} - 2}w_{11} + \frac{4}{l_{21} - 2}w_{12},$$

which implies

$$l_{11}w_{11} + l_{12}w_{12} < \frac{l_{21} + 2}{l_{21} - 2}w_{11} + \frac{l_{21} + 2}{l_{21} - 2}w_{12}$$

and in particular

$$l_{12} < \frac{l_{21} + 2}{l_{21} - 2}.$$

\square

Remark 5.10. Let $X = X(A, P)$ be as in Setting 5.4. For $c > 0$ the assumption $h(g_2) > -c$ leads to

$$-c - \frac{l_{12}}{l_{21}}(c + d_{221}) < d_{212} < 0.$$

Remark 5.11. Let $X = X(A, P)$ be as in Setting 5.4. If $h(g_1) < 1$ holds, then we have

$$-\frac{l_{11}}{l_{21}}d_{221} < d_{211} < -\frac{l_{11}}{l_{21}}d_{221} + 1 + \frac{l_{11}}{l_{21}}.$$

Lemma 5.12. *Let $X = X(A, P)$ be as in Setting 5.4. Assume $l_{21} \geq 3$. If $h(g_1) < 1$ and $h(g_2) > -2$ hold, then we have*

$$l_{11} < 2\frac{l_{21}}{l_{21} - 2}.$$

This bounds l_{11} in terms of l_{21} in the case $h(g_1) < 1$ and $h(g_2) > -2$. In particular, we then have $l_{11} \leq 5$ and we have $l_{11} \leq 2$ as soon as $l_{21} \geq 6$.

Proof. Observe that $w_{01} + w_{02} = l_{11}w_{11} + l_{12}w_{12}$. Thus, the third and the fourth inequalities of Lemma 5.8 give us the condition

$$l_{11}w_{11} + l_{12}w_{12} < 2w_{11} + 2w_{12} + 2w_{21}.$$

We arrive at the assertion by writing this out and estimating d_{212} as well as d_{211} according to Remarks 5.10 and 5.11. \square

Lemma 5.13. *Let $X = X(A, P)$ be as in Setting 5.4. Suppose that $h(g_1) < 1$ and $h(g_2) \leq -c$ holds for some $c \in \mathbb{Z}_{\geq 2}$. Then we have $l_{12} = 1$ and moreover*

$$\frac{l_{11}l_{21}}{l_{11} + l_{21}} < \frac{c+1}{c-1} - \frac{2}{c-1} \cdot \frac{l_{21}}{1+l_{21}}.$$

Proof. Since $h(g_2) \leq -1$ holds, we must have $|g_2| < 1$ and thus obtain $l_{12} = 1$. The line segment $A_{X,0}^c \cap \{y = -1\}$ is of length strictly smaller than 1 and $A_{X,0}^c \cap \{y = -c\}$ is of length at least $|g_2|$. Since $h(g_1) < 1$ holds, we conclude

$$\frac{l_{11}l_{21}}{l_{11} + l_{21}} = |g_1| < \frac{1 - |g_2|}{c-1}(1 + h(g_1)) + 1 < \frac{c+1}{c-1} - \frac{2}{c-1} \cdot \frac{l_{21}}{1+l_{21}}.$$

□

Remark 5.14. Let $X = X(A, P)$ be as in Setting 5.4. Assume $l_{12} = 1$ and $d_{112} = d_{212} = 0$. Then $w_{11} > 0$ and $w_{12} > 0$ imply

$$0 < d_{211}, \quad -\frac{l_{21}}{l_{11}}d_{211} < d_{221} < 0.$$

Moreover, the conditions $h(g_1) < 1$ and $h(g_2) > -c$ are equivalent to the following conditions

$$d_{211} < -\frac{l_{11}}{l_{21}}(d_{221} - 1) + 1, \quad d_{221} > -c(l_{21} + 1).$$

Lemma 5.15. *Let $X = X(A, P)$ be as in Setting 5.4. Suppose that $h(g_1) \geq 1$ holds. Then either $l_{11} = l_{12} = 1$ or $l_{11} = l_{21} = 2$ hold.*

Proof. First observe that in this case, the segment $A_{X,0}^c \cap \{y = 1\}$ can be of length at most 1, because otherwise we have lattice points different from the origin and the vertices in A_X^c . This means $l_{11} = 1$ or $l_{11} = l_{21} = 2$. □

A second series of estimates makes use of the whole anticanonical complex A_X^c . The strategy is to detect via A_X^c suitable three-dimensional lattice simplices with precisely one interior lattice point and to use the volume bounds given in [2] in order to control the entries of the defining matrix P . We will distinguish several cases, using the notation of Remark 5.7.

Proposition 5.16. *Let $X = X(A, P)$ be as in Setting 5.4. Suppose $l_{11} = l_{12} = 1$. Then we achieve by admissible operations $d_{112} = d_{212} = 0$ and obtain the estimates*

$$\begin{aligned} 3 &\leq (l_{21} + 1)d_{211} \leq 72, & 0 &\leq d_{111} < d_{211}, \\ -d_{211}l_{21} &< d_{221} < 0, & \frac{d_{111}d_{221}}{d_{211}} - l_{21} &< d_{121} < 0. \end{aligned}$$

Proof. Consider the convex hull C' of $A_{X,0}^c$ and v_{21} . We may regard C' as a polytope in \mathbb{Q}^3 by omitting the first coordinate. Then C' is contained in the polytope C with the vertices

$$(l_{21}, d_{121}, d_{221}), \quad (-1, d_{111}, d_{211}), \quad (-1, 1 + d_{111}, d_{211}), \quad (-1, 0, 0), \quad (-1, 1, 0).$$

Now, C is a lattice polytope having $(0, 0, 0)$ as the only interior lattice point. There are precisely two ways to write C as a union of two simplices,

$$C = C_1 \cup C_2 = C_3 \cup C_4.$$

For each of these simplices, the volume is $\text{vol}(C_j) = (l_{21} + 1)d_{211}/6$. If the origin lies in the interior of one of the C_j , then, according to [2, Thm. 2.2], its volume is at most 12. This gives the bound

$$(l_{21} + 1)d_{211} = 6 \cdot \text{vol}(C_j) \leq 72.$$

The remaining estimates follow from positivity of the weights w_{ij} . If the origin lies in $C_1 \cap C_2 \cap C_3 \cap C_4$, then we must have

$$l_{21} = 2, \quad d_{211} = -d_{221}, \quad d_{111} = -1 - d_{121}.$$

Positivity of the weights provides the inequalities $d_{121}, d_{221} < 0$. Since the origin is the only lattice point in $A_{X,0}^c$, we get $d_{121} > -5$ and $d_{221} > 3(d_{121} + 1)$, which altogether fulfill the estimates of this proposition. \square

Proposition 5.17. *Let $X = X(A, P)$ be as in Setting 5.4. Suppose $l_{21} = 2$.*

- (i) *If $h(g_1) < 1$ and $h(g_2) > -1$ hold, turn P by means of admissible operations into the shape of Remark 5.5. Then we are in one of the situations:*
 - (a) $d_{121} = 1, d_{221} = 0, (2 + l_{12})d_{211} + (2 + l_{11})(-d_{212}) \leq 36$,
 - (b) $d_{121} = 0, d_{221} = 1, (l_{11} - l_{12}) + (2 + l_{12})d_{211} + (2 + l_{11})(-d_{212}) \leq 36$.*In both situations the remaining entries d_{111}, d_{112} are bounded according to Remark 5.5.*
- (ii) *If $h(g_1) < 1$ and $h(g_2) \leq -1$ hold, then we have $l_{12} = 1$. Moreover adjusting $d_{112} = d_{212} = 0$ by admissible operations, we arrive in one of the following three situations:*
 - (a) $l_{11} = 1$ holds and Proposition 5.16 applies.
 - (b) $l_{11} = 2$ holds and we have estimates

$$-6 \leq d_{221} \leq -3, \quad d_{211} = 1 - d_{221}.$$

- (c) $3 \leq l_{11} < 140$ holds and we have estimates

$$\frac{-5l_{11} + 2}{l_{11} - 2} < d_{221} \leq -3, \quad -\frac{l_{11}}{2}d_{221} < d_{211} < -\frac{l_{11}}{2}d_{221} + \frac{l_{11}}{2}.$$

In both cases (b) and (c), the remaining entries of the defining matrix P are bounded by

$$0 \leq d_{121} < -d_{221}, \quad \frac{d_{121}d_{211}}{d_{221}} + 2\frac{d_{211}}{d_{221}} < d_{111} < \frac{d_{121}d_{211}}{d_{221}}.$$

- (iii) *If $h(g_1) \geq 1$ holds, then we have $l_{11} = l_{12} = 1$ and Proposition 5.16 applies.*

Proof. We prove (i). First observe that Remark 5.5 yields $d_{221} \in \{0, 1\}$ because of $l_{21} = 2$. If $d_{221} = 1$ holds, then Remark 5.5 implies $d_{121} = 0$. If $d_{221} = 0$ holds, then we must have $d_{121} = 1$ because v_{21} is a primitive lattice point. This leads to cases (a) and (b) as the only possibilities. For the estimate of case (a), we look at the lattice simplex C_1 given in \mathbb{Q}^3 as the convex hull of the following points:

$$(l_{11}, d_{111}, d_{211}), \quad (l_{12}, d_{112}, d_{212}), \quad (-2, 1, 0), \quad (-2, 3, 0).$$

To obtain the estimate of case (b), we look at the lattice simplex C_2 in \mathbb{Q}^3 given as the convex hull of the following points:

$$(l_{11}, d_{111}, d_{211}), \quad (l_{12}, d_{112}, d_{212}), \quad (-2, 0, 1), \quad (-2, 2, 1).$$

For the volumes, we obtain in both cases $\text{vol}(C_i) = (w_{11} + w_{12} + w_{21})/3$. Now, put the leaf $A_X^c \cap \tau_1$ of the anticanonical complex into \mathbb{Q}^3 by removing the second coordinate (which always equals zero) from its points. For $a = 0, -1, -2$, consider

$$H_a^+ := \{(x, y, z); x \geq a\} \subseteq \mathbb{Q}^3, \quad H_a^0 := \{(x, y, z); x = a\} \subseteq \mathbb{Q}^3.$$

Then $C_i \cap H_0^+$ equals $A_X^c \cap \tau_1$ and H_0^0 cuts out the lineality part $A_{X,0}^c$. In particular, by terminality of X and Theorem 1.4, the intersection $C_i \cap H_0^+$ has no interior lattice point and inside $C_i \cap H_0^0$ the origin is the only lattice point. The intersection $C_1 \cap H_{-1}^0$ has the vertices

$$\left(-1, \frac{l_{11} + d_{111} + 1}{l_{11} + 2}, \frac{d_{211}}{l_{11} + 2}\right), \quad \left(-1, \frac{3l_{11} + d_{111} + 3}{l_{11} + 2}, \frac{d_{211}}{l_{11} + 2}\right),$$

$$\left(-1, \frac{l_{12} + d_{112} + 1}{l_{12} + 2}, \frac{d_{212}}{l_{12} + 2}\right), \quad \left(-1, \frac{3l_{12} + d_{112} + 3}{l_{12} + 2}, \frac{d_{212}}{l_{12} + 2}\right),$$

while the intersection $C_2 \cap H_{-1}^0$ has the vertices

$$\begin{aligned} &\left(-1, \frac{d_{111}}{l_{11} + 2}, \frac{l_{11} + d_{211} + 1}{l_{11} + 2}\right), \quad \left(-1, \frac{2l_{11} + d_{111} + 2}{l_{11} + 2}, \frac{l_{11} + d_{211} + 1}{l_{11} + 2}\right), \\ &\left(-1, \frac{d_{112}}{l_{12} + 2}, \frac{l_{12} + d_{212} + 1}{l_{12} + 2}\right), \quad \left(-1, \frac{2l_{12} + d_{112} + 2}{l_{12} + 2}, \frac{l_{12} + d_{212} + 1}{l_{12} + 2}\right). \end{aligned}$$

The inequalities $h(g_1) < 1$ and $h(g_2) > -1$ together with the positivity of the weights ensure that the points of $C_i \cap H_{-1}^0$ never have an integral z -value. We can conclude that $C_i \subseteq H_{-2}^+$ has the origin as its only interior lattice point. Applying the bound $\text{vol}(C_i) \leq 12$ from [2, Thm. 2.2] and writing down the involved weights explicitly we arrive at the assertion.

We turn to (ii). By Lemma 5.13 we have $l_{12} = 1$. By admissible operations, we achieve $d_{112} = d_{212} = 0$. If $l_{11} = 1$ holds, we can apply Proposition 5.16. Let $l_{11} \geq 2$. For $l_{11} \geq 3$, the positivity of the weights and the constraints on the heights together with suitable admissible operations lead to all the bounds for the d_{ijk} stated in (c) except for the lower bound on d_{221} . For that, observe that the segment $A_{X,0}^c \cap \{y = -1\}$ has to be of length strictly smaller than 1 and conclude

$$\frac{-5l_{11} + 2}{l_{11} - 2} < d_{221}.$$

The next step is to bound l_{11} . For this, we consider the simplex $D \subseteq \mathbb{Q}^3$ given as the convex hull of following points

$$(l_{11}, d_{111}, d_{211}), \quad (1, 0, 0), \quad (-2, d_{121}, d_{221}), \quad (-2, d_{121} + 2, d_{221}).$$

Now, put the leaf $A_X^c \cap \tau_1$ of the anticanonical complex into \mathbb{Q}^3 by removing the second coordinate (which always equals zero). With the same notation as in part (i) of the proof, we see that $D \cap H_0^+$ equals $A_X^c \cap \tau_1$ and H_0^0 cuts out the lineality part $A_{X,0}^c$. For $l_{11} \geq 10$ the only possible values for d_{221} are $-3, -4, -5$. Moreover we already have $0 \leq d_{121} < -d_{221}$. Thus the allowed pairs (d_{121}, d_{221}) are

$$\begin{aligned} &(0, -3), \quad (1, -3), \quad (2, -3), \quad (1, -4), \quad (3, -4), \\ &(0, -5), \quad (1, -5), \quad (2, -5), \quad (3, -5), \quad (4, -5). \end{aligned}$$

Actually all of them, except the fourth, the seventh and the eighth, already provide an inner lattice point in the lineality part. Going through the remaining three pairs we are able to determine the inner lattice points of D other than the origin. These points can now only lie in H_{-1}^0 . By finding a simplex with exactly one inner lattice point and using [2, Thm. 2.2] we obtain $l_{21} \leq 140$. Here we treat the pair $(1, -4)$ as an example, since it provides the worst estimate. In this case D has vertices

$$v_{11} = (l_{11}, d_{111}, d_{211}), \quad v_{12} = (1, 0, 0), \quad a_1 := (-2, 1, -4), \quad a_2 := (-2, 3, -4),$$

and we get $p := (-1, 1, -2)$ as only inner lattice point other than the origin. We define simplices $D_1 := \text{conv}(p, v_{11}, v_{12}, a_1)$ and $D_2 := \text{conv}(p, v_{11}, v_{12}, a_2)$. The origin lies in one of the two simplices D_i . Bounding their volumes by 12 according to [2, Thm. 2.2] we obtain $l_{11} \leq 70$ if $0 \in D_1^\circ$ and $l_{11} \leq 140$ if $0 \in D_2^\circ$. Note that the origin cannot lie in $D_1 \cap D_2 = \text{conv}(p, v_{11}, v_{12})$: we would have $d_{211} = -2d_{111}$, but then terminality would require $\gcd(d_{111}, d_{211}) = 1$, which in turn fixes $d_{111} = -1$ and $d_{211} = 2$. Now with the second estimate of (c) $l_{11} < 1$ must hold, a clear contradiction to $l_{11} \geq 2$.

Now we turn to the case $l_{11} = 2$ and prove the estimates of (b). Here u_1 and u_2 are half-integral points, therefore we have $h(g_1) = 1/2$, which implies $d_{211} + d_{221} = 1$. The constraint $h(g_2) \leq -1$ is equivalent to $d_{221} \leq -3$. Estimates on d_{111} and d_{121} are found by positivity of the weights and admissible operations. For the lower bound on d_{221} we note that u_3 lies under the bisection of the fourth quadrant.

Requiring that no lattice point lies in $A_{X,0}^c$ except for the origin only leaves a confined area to place g_2 , namely $h(g_2) \geq -2$ must hold. This provides the bound $d_{221} \geq -6$.

Let us verify (iii). By Lemma 5.15 we have $(l_{11}, l_{12}) \in \{(1, 1), (2, 1), (2, 2)\}$. If both exponents are equal 1, then Proposition 5.16 applies straightforward. If both exponents are equal 2, then $|g_1| = |g_2| = 1$. This implies that the segment $A_{X,0}^c \cap \{y = 1\}$ is of length one and hence contains at least one lattice point. Lastly we show that the case $(l_{11}, l_{12}) = (2, 1)$ is also not possible. Here it holds $|g_1| = 1$ and two of the vertices are

$$u_1 = \left(\frac{1}{2}d_{111} + \frac{1}{2}d_{121}, \frac{1}{2}d_{211} + \frac{1}{2}d_{221}\right), \quad u_2 = u_1 + (1, 0).$$

We assume $h(g_1)$ to be non-integral, otherwise we would have a lattice point on g_1 itself. Nonetheless an integral point p is always in the lineality part, precisely at the height $h(g_1) - 1/2$ and it can be given explicitly as $p := \alpha u_1 + \beta u_2$ where

$$\alpha := -k - \frac{d_{111} + d_{121} + 2}{2(d_{211} + d_{221})}, \quad \beta := 1 + k + \frac{d_{111} + d_{121}}{2(d_{211} + d_{221})}$$

for an appropriate $k \in \mathbb{Z}_{\geq 0}$ that makes $0 \leq \alpha, \beta < 1$. Then we have

$$p = \left(\frac{1}{2}d_{111} + \frac{1}{2}d_{121} + k + 1, h(g_1) - \frac{1}{2}\right),$$

which is an integral point since we can always assume d_{111} and d_{121} to have the same parity. \square

Proposition 5.18. *Let $X = X(A, P)$ be as in Setting 5.4. Suppose $l_{21} \geq 3$.*

(i) *If $h(g_1) < 1$ and $h(g_2) > -2$ hold, then we are in one of the following three situations:*

(a) *We have $3 \leq l_{21} \leq 5$ and the other l_{ij} are bounded according to the table*

l_{21}	3	4	5
l_{12}	≤ 4	≤ 2	≤ 2
l_{11}	≤ 5	≤ 3	≤ 2

In this case turn P by means of admissible operations into the shape of Remark 5.5. Then we have $0 \leq d_{121}, d_{221} < l_{21}$ and the estimates

$$-2 - \frac{l_{12}}{l_{21}}(d_{221} + 2) < d_{212} < 0, \quad -\frac{l_{11}}{l_{21}}d_{221} < d_{211} < -\frac{l_{11}}{l_{21}}d_{221} + 1 + \frac{l_{11}}{l_{21}}$$

and the remaining two entries d_{111}, d_{112} are bounded according to Remark 5.5.

(b) *We have $6 \leq l_{21}$ and $l_{11} = l_{12} = 1$. Then all entries d_{ijk} can be bounded according to Proposition 5.16.*

(c) *We have $6 \leq l_{21}$, $l_{11} = 2$ and $l_{12} = 1$. Then we achieve $d_{112} = d_{212} = 0$ by suitable admissible operations and values and bounds for the remaining entries are given by the table*

d_{111}	0	1	1	1	1	2	2
d_{211}	1	2	3	4	5	3	5
l_{21}	≤ 141	≤ 71	≤ 71	≤ 179	≤ 177	≤ 137	≤ 143

and by the estimates

$$-2(l_{21} + 1) < d_{221} < 0, \quad \frac{d_{111}d_{221}}{d_{211}} - l_{21} < d_{121} < \frac{d_{111}d_{221}}{d_{211}}.$$

(ii) *If $h(g_1) < 1$ and $h(g_2) \leq -2$ hold, then we are in one of the following two situations:*

(a) *We have $l_{11} = l_{12} = 1$. Then l_{21} and the entries d_{ijk} can be bounded according to Proposition 5.16.*

- (b) We have $l_{11} = 2$, $l_{12} = 1$ and $l_{21} = 3, 4$. Then we achieve $d_{112} = d_{212} = 0$ by admissible operations and obtain the following estimates

$$-4(l_{21} + 1) < d_{221} < 0, \quad -\frac{2}{l_{21}}d_{221} < d_{211} < -\frac{2}{l_{21}}(d_{221} - 1) + 1,$$

$$0 \leq d_{121} < -d_{221}, \quad \frac{d_{211}(d_{121} + l_{21})}{d_{221}} < d_{111} < \frac{d_{211}d_{121}}{d_{221}}.$$

- (iii) If $h(g_1) \geq 1$ holds, then we have $l_{11} = l_{12} = 1$ and Proposition 5.16 applies.

Proof. Let us verify (i). Lemmas 5.9 and 5.12 provide us bounds on l_{11} and l_{12} in terms of l_{21} , namely those from the table of case (a) if $l_{21} < 6$, otherwise $l_{11} = 1, 2$ and $l_{12} = 1$. The other estimates of case (a) follow directly from Remarks 5.5, 5.10 and 5.11. From now on we have $l_{21} \geq 6$ and $l_{12} = 1$, thus we assume $d_{112} = d_{212} = 0$ by admissible operations. If $l_{11} = 1$ then we are in case (b) and Proposition 5.16 applies. If $l_{11} = 2$ then we have to prove the estimates of case (c). Writing down explicitly the inequalities $h(g_1) < 1$ and $h(g_2) > -2$ as well as the positivity of the weights already gives us the bounds for d_{121} and d_{221} and the following estimates

$$-\frac{2d_{221}}{l_{21}} < d_{211} < -\frac{2d_{221}}{l_{21}} + \frac{l_{21} + 2}{l_{21}}, \quad 0 \leq d_{111} < d_{211}.$$

All we are left to find is an upper bound for l_{21} . Note that by substituting the lower estimate for d_{221} in the upper estimate for d_{211} one obtains

$$0 < d_{211} < 5 + \frac{6}{l_{21}} \leq 6.$$

Thus we have a finite range (independent from l_{21}) for d_{211} and therefore for d_{111} too, namely

$$d_{111} \in \{0, 1, 2\}, \quad d_{111} < d_{211} \leq 5.$$

The cases $d_{111} = 3, 4$ are discharged, because there the origin lies outside of the lineality part $A_{X,0}^c$. Moreover, if $d_{111} = 0$ holds, then $d_{211} = 1$ must hold because of terminality. We look at the lattice polytope C in \mathbb{Q}^3 given as the convex hull of the following points:

$$(l_{21}, d_{121}, d_{221}), \quad (-2, d_{111}, d_{211}), \quad (-2, d_{111} + 2, d_{211}), \quad (-1, 0, 0), \quad (-1, 1, 0).$$

Now, put the leaf $A_X^c \cap \tau_2$ of the anticanonical complex into \mathbb{Q}^3 by removing the first coordinate (which always equals zero) from its points. For $a = 0, -1, -2$, consider

$$H_a^+ := \{(x, y, z); x \geq a\} \subseteq \mathbb{Q}^3, \quad H_a^0 := \{(x, y, z); x = a\} \subseteq \mathbb{Q}^3.$$

Then $C \cap H_0^+$ equals $A_X^c \cap \tau_2$ and H_0^0 cuts out the lineality part $A_{X,0}^c$. In particular, by terminality of X and Theorem 1.4, the intersection $C \cap H_0^+$ has no interior lattice point and inside $C \cap H_0^0$ the origin is the only lattice point. Other interior lattice points of C may only appear in $C \cap H_{-1}^0$. For any given pair (d_{111}, d_{112}) out of the finite set of possible pairs we find a simplex $B \subseteq C$ containing exactly one interior lattice point and bound its volume using [2, Thm. 2.2]. This technique is the same as the one used in the proof of the previous Proposition. This allows to bound l_{21} according to the table of case (c).

Now we prove (ii). By Lemma 5.13 we have $l_{12}=1$, therefore we can always achieve $d_{112} = d_{212} = 0$ by admissible operations. The same Lemma gives us $l_{11} = 1$ if $l_{21} \geq 5$ or if $h(g_2) \leq -4$. This case is covered by Proposition 5.16. Let us therefore assume $l_{21} \in \{3, 4\}$ and $h(g_2) > -4$, together with $l_{11} > 1$. Then Lemma 5.13 implies $l_{11} = 2$. Moreover Remark 5.14 provides estimates on d_{211} and d_{221} in terms of l_{21} . The last bounds on d_{111} and d_{121} are obtained as in Remark 5.5.

Lastly we turn to (iii). We have $l_{21} \geq 3$ and $h(g_1) \geq 1$. Then by Lemma 5.15 we must have $l_{11} = l_{12} = 1$. Hence Proposition 5.16 applies. \square

Concerning the remaining cases of Lemma 5.2, one shows with arguments similar to those used for Proposition 5.3 that (iii), (v), (vii) and (viii) do not provide terminal varieties. For the cases (ii), (iv) and (vi) we state without proof the bounds we obtained; as mentioned, the arguments are similar as in case (i) and will be presented elsewhere.

Proposition 5.19. *Let $X = X(A, P)$ be a non-toric terminal \mathbb{Q} -factorial Fano threefold of Picard number one such that P is irredundant and we have $r = 2$, $m = 0$ and $n = 5$, where $n_0 = 3$, $n_1 = n_2 = 1$. Then $l_{01} = l_{02} = l_{03} = 1$ hold and after suitable admissible operations the matrix P is of the form*

$$P = \begin{bmatrix} -1 & -1 & -1 & l_{11} & 0 \\ -1 & -1 & -1 & 0 & l_{21} \\ 0 & 1 & 0 & d_{111} & d_{121} \\ 0 & 0 & 1 & d_{211} & d_{221} \end{bmatrix},$$

where $l_{11} \geq l_{21}$ holds. In this setting, we have $2 \leq l_{21} \leq 5$ and we are left with the following situations:

- (i) We have $l_{21} = 2$. Then we achieve $d_{121} = 1$ by suitable admissible operations and we are in one of the following two cases:

- (a) $d_{221} = 0$, $l_{11} \leq 69$ hold and we have the estimates

$$-\frac{l_{11}}{2} - 1 < d_{211} < 0, \quad -l_{11} \leq d_{111} < -\frac{l_{11}}{2},$$

- (b) $d_{221} = 1$, $-35 \leq d_{111} < 0$ hold and we have the estimates

$$d_{111} \leq d_{211} < 0, \quad \max(2, -d_{111}) \leq l_{11} < -2d_{211}.$$

- (ii) We have $l_{21} = 3$. Then we achieve $0 \leq d_{121} \leq d_{221} < 3$ by suitable admissible operations and the value l_{11} is bounded according to the table

d_{121}	0	0	1	1	2
d_{221}	1	2	1	2	2
l_{11}	≤ 71	≤ 211	≤ 103	≤ 211	≤ 69

and for the remaining entries we obtain the estimates

$$-\frac{l_{11}}{3}(d_{121} + 1) - 1 < d_{111} < -\frac{l_{11}}{3}d_{121},$$

$$-\frac{l_{11}}{3}(d_{221} + 1) - 1 < d_{211} < -\frac{l_{11}}{3}d_{221}.$$

- (iii) We have $l_{21} = 4$ or $l_{21} = 5$. Then we have following estimates

$$l_{11} < \frac{3l_{21}}{l_{21} - 3}, \quad 0 \leq d_{121}, d_{221} < l_{21},$$

$$-\frac{l_{11}d_{121}}{l_{21}} - l_{11} < d_{111} < -\frac{l_{11}d_{121}}{l_{21}}, \quad -\frac{l_{11}d_{221}}{l_{21}} - l_{11} < d_{211} < -\frac{l_{11}d_{221}}{l_{21}}.$$

Proposition 5.20. *Let $X = X(A, P)$ be a non-toric terminal \mathbb{Q} -factorial Fano threefold of Picard number one such that P is irredundant and we have $r = 3$, $m = 0$ and $n = 6$, where $n_0 = n_1 = 2$, $n_2 = n_3 = 1$. Then $l_{01} = l_{02} = l_{11} = l_{12} = 1$ hold and after suitable admissible operations the matrix P is of the form*

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & l_{21} & 0 \\ -1 & -1 & 0 & 0 & 0 & l_{31} \\ 0 & 1 & d_{111} & 0 & d_{121} & d_{131} \\ 0 & 0 & d_{211} & 0 & d_{221} & d_{231} \end{bmatrix},$$

such that $l_{21} \geq l_{31}$ holds. In this setting $l_{31} = 2, 3$ holds and we are left with the following situations:

- (i) We have $l_{31} = 2$. Then we have $d_{211} = 1$ and we can achieve $d_{111} = 0$ by a suitable admissible operation. The other entries of P are then bounded according to the table

$$\begin{array}{c|c|c|c} d_{131} & 0 & 1 & 1 \\ d_{231} & 1 & 0 & 1 \\ \hline l_{21} & \leq 33 & \leq 141 & \leq 69 \end{array}$$

and the estimates

$$-\frac{l_{21}}{2}(d_{231} + 1) - 1 < d_{221} < -\frac{l_{21}}{2}d_{231},$$

$$-\frac{d_{221}}{l_{21}} - \frac{d_{231}}{2} < d_{211} < -\frac{d_{221}}{l_{21}} - \frac{d_{231}}{2} + \frac{2 + l_{21}}{2l_{21}}.$$

- (ii) We have $l_{31} = 3$. Then $3 \leq l_{21} \leq 5$ holds, we obtain $0 \leq d_{131}, d_{231} < 3$ and we have the estimates

$$-\frac{l_{21}}{3}(d_{231} + 1) - 1 < d_{221} < -\frac{l_{21}}{3}d_{231},$$

$$-\frac{d_{221}}{l_{21}} - \frac{d_{231}}{3} < d_{211} < -\frac{d_{221}}{l_{21}} - \frac{d_{231}}{3} + \frac{3 + l_{21}}{3l_{21}},$$

$$0 \leq d_{111} < d_{211}l_{31}$$

$$\frac{d_{111}(l_{21}d_{231} + 3d_{221}) - l_{21}d_{211}d_{131}}{3d_{211}} - l_{21} < d_{121} < \frac{d_{111}(l_{21}d_{231} + 3d_{221}) - l_{21}d_{211}d_{131}}{3d_{211}}.$$

Proposition 5.21. Let $X = X(A, P)$ be a non-toric terminal \mathbb{Q} -factorial Fano threefold of Picard number one such that P is irredundant and we have $r = 2$, $m = 1$ and $n = 4$, where $n_0 = 2$, $n_1 = n_2 = 1$. Then $l_{01} = l_{02} = 1$ hold and after suitable admissible operations the matrix P is of the form

$$P = \begin{bmatrix} -1 & -1 & l_{11} & 0 & 0 \\ -1 & -1 & 0 & l_{21} & 0 \\ 0 & 1 & d_{111} & d_{121} & 0 \\ 0 & 0 & d_{211} & d_{221} & 1 \end{bmatrix},$$

where $l_{11} \geq l_{21}$ and $0 \leq d_{121}, d_{221} < l_{21}$ hold. In this situation, one has the estimates

$$-\frac{l_{11}}{l_{21}}d_{121} - l_{21} < d_{111} < -\frac{l_{11}}{l_{21}}d_{121}, \quad -\frac{l_{11}}{l_{21}}(d_{221} + 1) - 1 < d_{211} < -\frac{l_{11}}{l_{21}}d_{221}.$$

Moreover, $l_{21} \leq 7$ holds and l_{11} is bounded according to the table

$$\begin{array}{c|c|c|c|c|c|c} l_{21} & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline l_{11} & \leq 140 & \leq 211 & \leq 283 & \leq 19 & \leq 11 & \leq 9 \end{array}.$$

Remark 5.22. Propositions 5.16 to 5.21 give us effective bounds on the entries of the defining matrices P for the terminal \mathbb{Q} -factorial Fano threefolds $X = X(A, P)$ with effective two-torus action and Picard number $\rho(X) = 1$. In order to prove Theorem 1.1 one still has to figure out the terminal ones among all candidates $X = X(A, P)$, where P fulfills these bounds. This means to check Condition 1.4 (v); we do it by computer, using [6] where the anticanonical complex A_X^\vee is implemented. Using the explicit knowledge of canonical Fano tetrahedra provided by Kasprzyk's classification [13], one can reduce the number of testing cases and obtains more specific bounds in the remaining cases.

Remark 5.23. If one adds the assumption “ $\text{Cl}(X)$ finitely generated” in Theorem 1.1, then, without further changes, all the results and proofs of the paper are valid over any algebraically closed field of characteristic zero.

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